VIRTUAL CROSSINGS AND FILTRATIONS IN LINK HOMOLOGY

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A dissertation submitted to the faculty at the University of North Carolina at Chapel Hill in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Mathematics.

Chapel Hill
2014

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ABSTRACT

Michael A. Abel: Virtual Crossings and Filtrations in Link Homology
(Under the direction of Lev Rozansky)

In 2006 Khovanov and Rozansky introduced a triply-graded link homology theory categorifying the HOMFLY-PT polynomial [24]. Khovanov later gave an alternate construction of HOMFLY-PT homology using Rouquier's braid group action on the category of Soergel bimodules [21, 38, 39]. Soergel bimodules can be filtered by submodules which are the images of virtual crossings in an action of the virtual braid group on the category of graded $\mathbb{Q}[x_1, ..., x_n]$-bimodules. We conjecture that this filtration extends to HOMFLY-PT homology. We prove that the filtered version of HOMFLY-PT homology is invariant under Reidemeister I and II moves, and conjecture that Reidemeister III does the same. We show that Reidemeister III can violate filtration by at most two levels. This filtration would give a fourth grading on HOMFLY-PT homology, which has been suggested by experimental calculations in recent physics research. The use of filtrations allows us to replace proofs done by generators and relations for Soergel bimodules with more intuitive and diagrammatic proofs.

ACKNOWLEDGMENTS

I would like to thank the mathematics community at the University of North Carolina at Chapel Hill. In particular, I appreciate the useful questions and conversations with my committee members: Prakash Belkale, Shrawan Kumar, Justin Sawon and Jonathan Wahl. I am especially grateful to my advisor, Lev Rozansky, for his incredible guidance, patience, and support. I would like to thank Eugene Gorsky, Mikhail Khovanov, and Robert Lipshitz for the many enlightening conversations I have had with them while working on this project. I would also like to give thanks to my fellow graduate students Alex, Amanda, Jake, Justin, Merrick, Sam and Ryan for their support, many wonderful conversations, and their friendship.

This work would not have been possible without the support of my family. I cannot thank Jackie enough for her support, encouragement and companionship throughout this long process. Finally I want to thank my mentor and great friend, Dave Smith, for guiding me toward studying mathematics.

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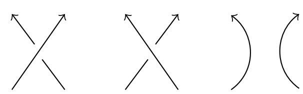
CHAPTER 1: Introduction

Let L and L' be links embedded in $\mathbb{R}^3$. A main goal in knot theory is to determine if L and L' are in the same isotopy class of links. A more specific question to ask is how to tell if a link L is isotopic to the unlink. A main tool knot theorists use to try to answer these questions are polynomial knot invariants. Examples of polynomial knot invariants are the Alexander, Jones, and the HOMFLY-PT polynomials. These polynomials can all be expressed using a value for the unknot and skein relations, or relations of the form $aL_- + bL_+ = cL_0$ for some a, b and c, where L_-, L_+ and L_0 are links the same except in a neighborhood of a point where they are shown in Figure 1.1.



L_+ L_- L_0

Figure 1.1: Conway Triple

In particular the HOMFLY-PT polynomial, $P(L) \in \mathbb{Z}(q, a)$ is given by the skein relation $aL_+ - a^{-1}L_- = (q - q^{-1})L_0$ with

$$P(Unknot) = \frac{a - a^{-1}}{q - q^{-1}}.$$

In this work we recall the categorification of the HOMFLY-PT polynomial given by Khovanov and Rozansky. Then we introduce a filtration on the complex associated to a link, $\mathcal{G}(L)$, and develop a framework in which we represent these filtrations in terms of convolutions and virtual crossings. Then we will prove the invariance of this filtration under Reidemeister moves I and II and conjecture invariance under Reidemeister III. If our conjecture holds, then we will have constructed a quadruply-graded link homology theory as conjectured by Gorsky, Gukov, and Stosic [16]. Finally we will give bounds on the possible error from Reidemeister III and provide some computations of the filtered HOMFLY-PT homology for $(2, n)$-torus links.

1.1 Historical Context

Khovanov in 2000 introduced the idea of categorification to knot theory in the form of what is now known as Khovanov homology [20]. The term categorification, first introduced by Crane and Frenkel [11], refers to a process in which we construct a category $\mathcal{C}$ whose Grothendieck group $K_0(\mathcal{C})$ is the algebraic structure A we wished to categorify. In particular for homological link invariants we wish to find an object X in a new category $\mathcal{C}$ whose image in the Groethendieck group, $[X] \in K_0(\mathcal{C})$, is the original polynomial link invariant $P(L) \in A$.

Khovanov homology was first constructed as a categorification of the Jones polynomial, lifting the skein relation $q^2 L_+ - q^{-2} L_- = (q - q^{-1}) L_0$ to a long exact sequence of homology groups. Not only is this homology theory computable and accessible [3, 5], but it has been shown to be a stronger knot invariant than the Jones polynomial [3], functorial with respect to link cobordisms [9], and an unknot detector [26]. Also more classical topological interpretations have been given, such as Rasmussen's combinatorial proof of the Milnor conjecture [36]. Until Rasmussen's proof, the only proof of the Milnor conjecture used gauge theory which was discovered by Kronheimer and Mrowka [25].

In 2004, Khovanov and Rozansky introduced a categorification of the $\mathrm{SL}(n)$ link invariant using $\mathbb{Z}_2$-graded matrix factorizations [23]. This was specifically a categorification of the MOY construction of the $\mathrm{SL}(n)$ polynomial using directed weighted graphs representing intertwiners between tensor powers of fundamental representations of $\mathcal{U}_q(\mathfrak{sl}_n)$ [31]. Khovanov-Rozansky $\mathrm{SL}(n)$ homology is also functorial (possibly up to a sign). In 2005, Khovanov and Rozansky introduced a deformation of Khovanov-Rozansky homology which categorfied the entire HOMFLY-PT polynomial as a triply-graded homology theory [24]. Unfortunately this theory is not as well-behaved as the previous theories mentioned. It must be constructed from the closure of a braid instead of being able to define it directly from a link diagram. However it was shown by Rasmussen that there exist spectral sequences from the HOMFLY-PT homology to the $\mathrm{SL}(n)$ homology [35].

In 2006 Khovanov showed that HOMFLY-PT homology was related to Rouquier's action of the braid group, $\mathcal{B}_n$, on the homotopy category of Soergel bimodules [21, 38]. In particular if you construct the Rouquier complex of a braid β, apply the Hochschild homology functor $\mathrm{HH}(\bullet, R)$ to each chain module, and then compute the homology of the resulting complex, you have a link homology theory. This link homology is isomorphic to the HOMFLY-PT homology which Khovanov and Rozanksy had previously constructed.

In [22] Khovanov and Rozansky give a conjectural categorification for the Kauffman SO($2n$) polynomial. This categorification takes deformations of Soergel bimodules of type D_n into matrix factorizations, and resolves them as convolutions of twisted complexes of matrix factorizations associated to virtual crossings. In an appendix of [22], Khovanov and Rozansky showed how this idea can be applied to the case of SL(n) homology and HOMFLY-PT homology as well. In particular, in the HOMFLY-PT case, these convolutions of virtual crossings contain well-known information associated to filtrations of Soergel bimodules by their irreducible submodules.

Soergel bimodules were first introduced by Wolfgang Soergel in [39] to give an alternate proof of the Kazhdan-Lusztig positivity conjecture in a purely algebraic manner. Soergel bimodules give a categorification of the Kazhdan-Lusztig basis of the Hecke algebra H_n. Precisely, let $\mathcal{S}$ be the full subcategory of Soergel bimodules in the category of $\mathbb{Q}[x_1, ..., x_n]$-bimodules, then $K_0(\mathcal{S}) \cong H_n$. If B_i is the Soergel bimodule associated to the simple transposition $s_i = (i \; i+1)$, then $[B_i] = b_i$ where b_i is the Kazhdan-Lusztig basis element associated to the simple transposition s_i.

In his alternate proof of the Kazhdan-Lusztig positivity conjecture, Soergel used filtrations on Soergel bimodules and proved that the associated graded bimodules contain the information needed to compute Kazhdan-Lusztig polynomials. The irreducible submodules, also called *standard bimodules*, can be used to extend Rouquier's braid group action to an action of the virtual braid group in the homotopy category of $\mathbb{Q}[x_1, ..., x_n]$-bimodules. Thiel explicitly showed how to do this in terms of Rouquier complexes in [42].

There have been results in recent years that show there exists a deeper structure inside HOMFLY-PT homology. Webster and Williamson has proven that a construction of HOMFLY-PT homology exists in terms of the cohomology of various sheafs of certain algebraic groups [44]. They were able to give a construction of colored HOMFLY-PT homology, a homology theory which categorifies the colored HOMFLY-PT polynomial, in terms of sheaf cohomology and algebraic groups as well. Also in recent work by Oblomkov, Rasmussen, and Shende it has been conjectured that the HOMFLY-PT homology of a link of a curve singularity can be described in terms of weight polynomials of Hilbert schemes of points scheme-theoretically supported on the singularity [32].

Gorsky believes that this approach gives evidence for a quadruply-graded theory for HOMFLY-

PT homology [15]. In particular, we associate to a link a special $\mathbb{C}^* \times \mathbb{C}^*$-equivariant sheaf $\mathcal{E}_L$ on the Hilbert scheme of n points on $\mathbb{C}^2$. Two of the gradings corresponds to the $\mathbb{C}^* \times \mathbb{C}^*$-action on $\mathcal{E}_L$. The Hochschild grading is recovered by studying the sheaf $\operatorname{Ext}(\mathcal{E}_L, \mathcal{F}_j)$ for some special sheafs $\mathcal{F}_j$. A new fourth grading can be found by looking at the higher Ext sheafs. This is precisely the grading we are trying to recover.

1.2 Results and Methods

In Chapters 2 and 3 we give a quick overview of basic knot theory and its connection to the study of braid groups, the HOMFLY-PT polynomial, the MOY construction of the HOMFLY-PT polynomial, Soergel bimodules, and HOMFLY-PT homology. In Chapter 4 we introduce the notions of virtual knot theory and describe Soergel bimodules in terms of standard bimodules. As mentioned before, Thiel showed that these standard bimodules are the images of virtual crossings in an action of the virtual braid group.

Let $R = \mathbb{Q}[x_1, ..., x_n]$, B_i be the Soergel bimodule associated to the transposition s_i, and let R_i be the R-bimodule with the action $x_j \cdot f = x_j f$, $f \cdot x_j = x_{s_i(j)} k$. Then if we consider these bimodules as objects in $\mathcal{D}^b(\mathcal{A})$, the bounded derived category of R-bimodules, we can give B_i as a mapping cone in two ways as shown in Figure 1.2. We have to pass to $\mathcal{D}^b(\mathcal{A})$ because these maps, S and S', are actually elements of $\operatorname{Ext}^1(R, R_i)$ and $\operatorname{Ext}^1(R_i, R)$ respectively. Here we label the diagrams with their corresponding bimodules and use a box to denote a mapping cone. We will explain these diagrammatics in more detail in Chapters 3 and 4.

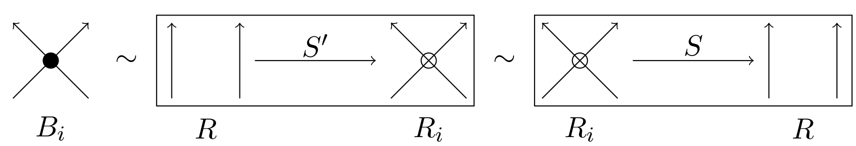


Figure 1.2: B_i as a mapping cone in $\mathcal{D}^b(\mathcal{A})$.

These mapping cones correspond to two non-equivalent filtrations of B_i by standard bimodules, we will denote them by B_i^- and B_i^+ respectively. The first has R_i as a submodule and R as a quotient module, and the second has R as a submodule and R_i as a quotient module. More importantly, the mapping cones are comprised of the associated graded bimodules of B_i. We can then construct mapping cone presentations for Bott-Samelson bimodules as well. Let $w = s_{i_1} \cdots s_{i_k}$ be a non-reduced word in S_n, then a Bott-Samelson bimodule is a bimodule of the form $BS(w) = B_{i_1} \otimes_R \cdots \otimes_R B_{i_k}$. Direct computation gives us the following proposition.

Proposition. *$BS(w)$ is quasi-isomorphic to an iterated mapping cone formed by tensoring together the mapping cone presentations of B_{i_j} for all j. The filtration induced on $BS(w)$ from this iterated mapping cone is equivalent to the tensor product filtration.*

We can next prove the following relations for filtered Soergel bimodules in $\mathcal{D}^b(\mathcal{A})$. We will use the notation $\sim_F$ to mean filtered quasi-isomorphic.

Proposition (Filtered Far Commutativity)**.** *Let $\varepsilon = +$ or $-$ and let $\delta = +$ or $-$. Then $B_i^\varepsilon \otimes_R B_j^\delta \sim_F B_j^\delta \otimes_R B_i^\varepsilon$ for $|i-j| \geq 2$*

Proposition (Filtered MOY II)**.** *Let $\varepsilon = +$ or $-$ and let $\delta = +$ or $-$. Then*

$$B_i^\varepsilon \otimes_R B_i^\delta \sim_F B_i^c \oplus \mathbf{bq}^2 B_i^{c'},$$

where $c = +$ and $c' = -$ if $\varepsilon \neq \delta$ and $c = -$ and $c' = +$ if $\varepsilon = \delta$.

Proposition (Filtered MOY III)**.** *Let $\delta_1, \delta_2, \delta_3 \in \{+,-\}$. Then*

$$B_i^{\delta_1} \otimes_R B_{i+1}^{\delta_2} \otimes_R B_i^{\delta_3} \sim_F \begin{cases} Cone(Y \to \mathbf{bq}^2 B_i^{-\delta_1\delta_2\delta_3}) & \text{if } \delta_2 = + \\ Cone(\mathbf{bq}^2 B_i^{-\delta_1\delta_2\delta_3} \to Y) & \text{if } \delta_2 = - \end{cases}$$

Here Y is the quotient bimodule $(B_i^{\delta_1} \otimes_R B_{i+1}^{\delta_2} \otimes_R B_i^{\delta_3})/B_i^{-\delta_1\delta_2\delta_3}$. with the induced filtration.

In Chapter 5, we redefine the Rouquier complex in $\mathcal{K}om(\mathcal{D}^b(\mathcal{A}))$ as shown in Figure 1.3 after the appropriate grading shifts. Where χ_i and χ_o are the natural inclusion and projection maps associated to a mapping cone.

Given our new filtered Rouquier complex, we will prove the following results. Let $\mathcal{G}$ be the new Rouquier functor as above where $\mathcal{G}(\alpha\beta) = \mathcal{G}(\alpha) \otimes_R \mathcal{G}(\beta)$ and $\mathcal{G}(1) = R$. Let $\mathcal{H}_F(\beta)$ denote the HOMFLY-PT homology of the closure of the braid β with the induced filtration.

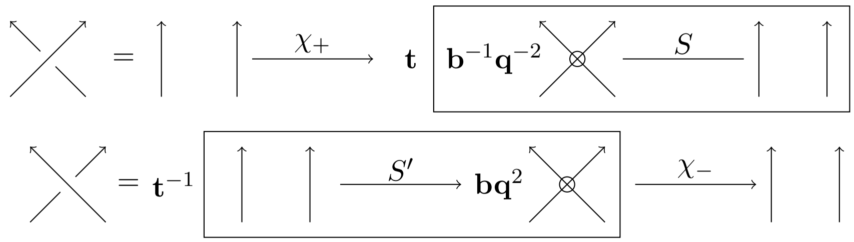


Figure 1.3: Redefinition of the Rouquier complex.

Theorem (Filtered Reidemeister II)**.** *The Reidemeister II isomorphism is a filtered homotopy equivalence. That is,*

$$\mathcal{G}(\sigma_i\sigma_i^{-1}) \simeq_F \mathcal{G}(1).$$

Theorem (Filtered Markov Moves)**.** *Let $\beta \in \mathcal{B}_n$. The following moves induce filtered isomorphisms on $\mathcal{H}_F(\beta)$.*

I. Braid conjugation on β, that is replacing $\beta = \alpha\alpha'$ with $\alpha'\alpha$.

II. Positive or negative (de)stabilization on β, that is doing the replacement $\beta \mapsto \beta\sigma_n^{\pm 1}$ or vice versa.

However, we can only conjecture invariance under Reidemeister III. We will also discuss this in more detail in Chapter 5.

Conjecture. *Let β and β' be two braid representatives of a link L which differ only by a Reidemeister III isotopy (that is the relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$). Then $\mathcal{H}_F(\beta)$ is filtered isomorphic to $\mathcal{H}_F(\beta')$.*

This conjecture and the preceding theorems immediately imply the following conjecture.

Conjecture. *Let L be a link and let β be a braid representative of L. Then $\mathcal{H}_F(\beta)$ is a filtered link invariant whose associated graded gives a quadruply-graded link homology theory.*

Though we cannot prove invariance Reidemeister III, we can prove the following statement giving a bound on the possible error given by Reidemeister III.

Theorem. *Let β be a braid representative of a link L. Let x be a generator of tri-degree (i, j, k) in $\mathcal{H}_F(\beta)$ with filtration degree m (That is, $x \in F_m(\mathcal{H}_F(\beta))$ but $x \notin F_{m+1}(\mathcal{H}_F(\beta))$). Suppose that β' is another braid representative of L differing from β by a Reidemesiter III isotopy. Then the*

Reidemeister III map from $\mathcal{H}_F(\beta)$ to $\mathcal{H}_F(\beta')$ changes the filtration degree of x by at most 2 and fixes all other gradings.

We now summarize the new process in contrast to the original process for HOMFLY-PT homology. Let $\mathcal{A}$ denote the category of graded R-bimodules. In the original process we apply the Rouquier functor, $\mathcal{F}$, to the braid β, then we apply the Hochschild homology functor to get a complex of doubly-graded vector spaces, then we apply the homology functor to get HOMFLY-PT homology.

In our new process we begin with a braid β, apply the filtered Rouquier functor $\mathcal{G}$ to get a complex of objects in $\mathcal{D}^b(\mathcal{A})$. We then apply the Hochschild homology functor to get a complex of doubly-graded filtered vector spaces. Finally we apply the homology functor to get a filtered triply-graded vector space, $\mathcal{H}_F(\beta)$. We can then take the associated graded of this vector space to get a quadruply-graded vector space. At every step we may apply a forgetful functor for to return to the original process. The diagram in Figure 1.4 shows this process.

$$\begin{tikzcd}
 & \mathcal{K}om(\mathcal{D}^b(\mathcal{A})) \arrow[r, "\mathrm{HH}(\bullet, R)"] \arrow[dd, "for"'] & \mathcal{K}om(F\mathbb{Q} - g_2mod) \arrow[r, "H"] \arrow[dd, "for"'] & F\mathbb{Q} - g_3mod \arrow[d, "G"'] \\
\mathcal{B}_n \arrow[ur, "\mathcal{G}"] \arrow[dr, "\mathcal{F}"'] & & & \mathbb{Q} - g_4mod \arrow[d, "for"'] \\
 & \mathcal{K}om(\mathcal{A}) \arrow[r, "\mathrm{HH}(\bullet, R)"'] & \mathcal{K}om(\mathbb{Q} - g_2mod) \arrow[r, "H"'] & \mathbb{Q} - g_3mod
\end{tikzcd}$$

Figure 1.4: The process of computing $\mathcal{H}(\beta)$ and $\mathcal{H}_F(\beta)$

In Chapter 6 we will finish the text with a discussion of future work with colored HOMFLY-PT homology and invariants of transverse links.

CHAPTER 2: The HOMFLY-PT and $SL(N)$ Polynomials

In this chapter we begin defining elementary concepts in knot theory, and its relationship to braid groups. We then give two different constructions of both the HOMFLY-PT and $SL(N)$ polynomials. Our first construction is the classical skein theoretic definition first given by Hoste, Ocneanu, Millet, Freyd, Lickorish, and Yetter [14] and also independently by Przytycki and Traczyk [34]. This definition arises by studying a representation of the braid group $\mathcal{B}_n$ on the Hecke Algebra $H_n(q, a)$. Also we will discuss the construction given by Murakami, Ohtsuki, and Yamada using colored trivalent graphs [31]. This approach uses the Kazdahn-Lusztig basis of $H_n(q, a)$ in defining the combinatorics of the trivalent graphs. The second construction is what is categorified in the subsequent link homology theories.

2.1 Preliminaries

Knot theory first appeared as a mathematical theory at the end of the 18th century. It was studied by A.T. Vandermonde, C.-F. Gauss, F. Klein and M. Dehn. A more systematic study of knots began at the end of the 19th century when mathematicans and physicists both began to tabulate knots. The preceding history and the following defintions are adapted from [28].

Definition 2.1.1. A *link* of m components is a subset of S^3, or of $\mathbb{R}^3$, that consists of m disjoint, smooth, simple closed curves. A link of one component is called a *knot*.

Intuitively we may envision a knot as a string which has been tangled up in some fashion and fused at the ends. With this perspective it makes sense to consider two knots the "same" if we may move around the strings without breaking them to make the two knots look identical. The following definition makes this idea mathematically precise.

Definition 2.1.2. Two links are called *isotopic* if one of them can be transformed to the other by a diffeomorphism h of S^3, or of $\mathbb{R}^3$, such that h is homotopic to the identity in the class of diffeomorphisms.

We will normally use the term link to mean both a link and its isotopy class. The main question in knot theory is the following: Given two links, L_1 and L_2, can we determine if L_1 is isotopic to L_2? The first step in attempting to solve these problems is to define a combinatorial method of describing links.

Definition 2.1.3. A *link diagram*, D, of a link, L, is a projection onto a copy of $\mathbb{R}^2$ embedded inside $\mathbb{R}^3$ that keeps the crossing information. In particular, the only singular points of the projection are transverse double points and the preimages are distinguished by their z-coordinates, where z is the coordinate transverse to the copy of $\mathbb{R}^2$.

A simpler, yet still very difficult, question is can we determine if a link is isotopic to the unlink? The *unlink* is a link of n components which has a link diagram consisting of n disjoint copies of S^1. A natural question can be asked. Is it possible to determine if two link diagrams D_1 and D_2 are diagrams for isotopic links? This question was answered positively by Reidemeister in 1926 [37] and by Alexander and Briggs in 1927 [2]

Theorem 2.1.4. *Suppose that D_1 and D_2 are two link diagrams. Then D_1 and D_2 represent isotopic links if they can be related by the a sequence of the following three moves, known as Reidemeister moves.*

I: Inserting or deleting a twist in either direction.

II: Move one strand completely on top of another.

III: Move a strand completely over or under another crossing.

These moves may be seen in Figure 2.1.

We may also study *oriented links*, that is a link where each component has a preferred direction which we will denote in a link diagram by the inclusion of arrows. If a link has n components then there are 2^n possible orientations. We will attempt to distinguish isotopy classes of links using *link invariants*. Roughly speaking, a link invariant is an object, $P(L)$ we assoicate to a link such that the following is true: If two links, L_1 and L_2, are isotopic then their link invariants, $P(L_1)$ and $P(L_2)$, are equivalent.

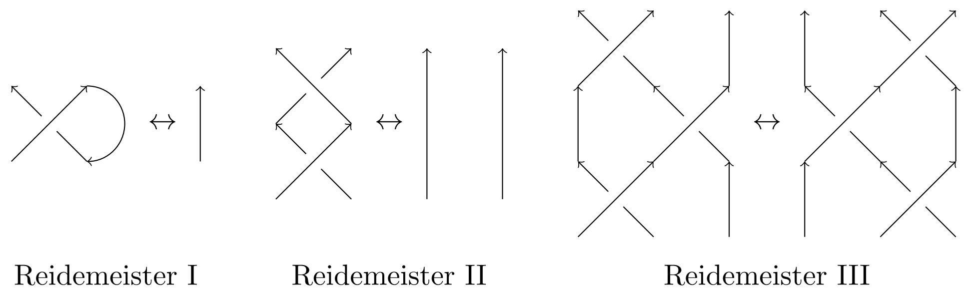


Figure 2.1: Reidemeister Moves

2.2 Braids and Oriented Links

Next we discuss braids and their relationship to oriented links.

Definition 2.2.1. A *braid of n strings*, or briefly an n-braid, is n disjoint oriented arcs traversing $D^2 \times [0,1]$ with strictly increasing $t \in [0,1]$. The arcs will join n standard fixed points in both $D^2 \times \{0\}$ and $D^2 \times \{1\}$. By convention we will assume that each arc's orientation is in the direction of increasing $t \in [0,1]$.

We can represent braids in diagrams in a similar manner to how we represent links. We will project D^2 onto an interval I in such a way that the n standard fixed points project onto distinct points. The we can easily see that two braids are isotopic if their diagrams differ by a sequence of Reidemeister moves of type II or III. Note that we may compose braids in the following manner. Suppose we have two n-braids β and β', then we define the product braid $\beta'\beta$ by juxteposing two braids and rescaling the t-coordinate as shown below in Figure 2.2.

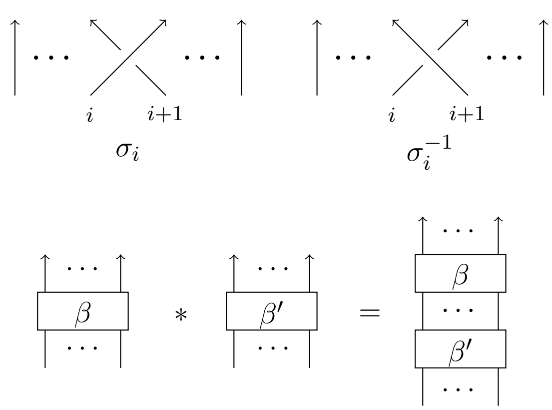


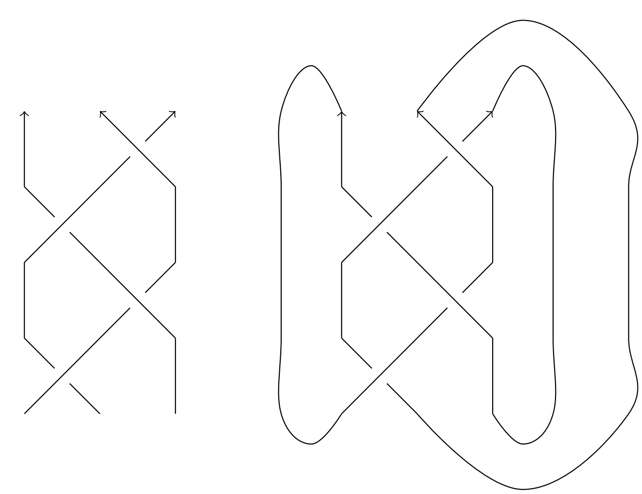
Figure 2.2: Elementary Braids and Braid Multiplication

Define the elementary n-braids $\sigma_1^{\pm 1}, ..., \sigma_{n-1}^{\pm 1}$ by the diagrams above in Figure 2.2. It is not hard to see that $\sigma_i \sigma_i^{-1}$ is isotopic to a braid with no crossings and $\sigma_i \sigma_{i+1} \sigma_i$ is isotopic to $\sigma_{i+1} \sigma_i \sigma_{i+1}$ using the Reidemeister moves. Therefore we may place a group structure on n-braids. We will associate to the braid with no crossings the identity element of the group.

Definition 2.2.2. The Artin n-braid group, which we denote by $\mathcal{B}_n$, is the group generated by the elementary n-braids with the following presentation:

$$\mathcal{B}_n = \langle \sigma_1, ..., \sigma_{n-1} | \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1},\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| \geq 2 \rangle.$$

We will use this group structure in the sequel when describing the HOMFLY-PT polynomial in terms of representations of $\mathcal{B}_n$. Now we want to look at the relationship between braids and links. Note that if we take a braid diagram for a braid β and identify the endpoints in the manner shown in Figure 2.3, we receive an oriented link in S^3. We will call this link the *closure* of β and denote it by $\bar{\beta}$. The first question we may ask is if any link can be constructed in such a manner. The answer was given by Alexander in 1923.



$\beta = \sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1}$ $\quad$ $\bar{\beta}$ = Figure-eight knot

Figure 2.3: A Braid and Its Closure as an Oriented Link

Theorem 2.2.3 (Alexander's Theorem [1])**.** *Let L be an oriented link. Then there exists a braid β such that $\bar{\beta}$ is isotopic to L.*

The next question we can ask is can we determine if two braids have isotopic closures. Markov in 1935 determined exactly when two braids give rise to isotopic links [30]. First we will define the two moves Markov introduced. These moves are pictured in Figure 2.4.

Definition 2.2.4. Let $\alpha, \beta \in \mathcal{B}_n$. A *Type I Markov move*, or *conjugation*, takes $\alpha\beta$ to $\beta\alpha$.

Definition 2.2.5. Let $\beta \in \mathcal{B}_n$. A *Type II Markov move*, or *(de)stabliazation*, takes $\beta \mapsto \beta\sigma_n^{\pm 1} \in \mathcal{B}_{n+1}$ (or vice versa).

Theorem 2.2.6 (Markov's Theorem)**.** *Let $\beta \in \mathcal{B}_n$ and $\beta' \in \mathcal{B}_{n'}$. β and β' have isotopic closures if and only if we may transform β to β' by a sequence of braid isotopies and Markov moves.*

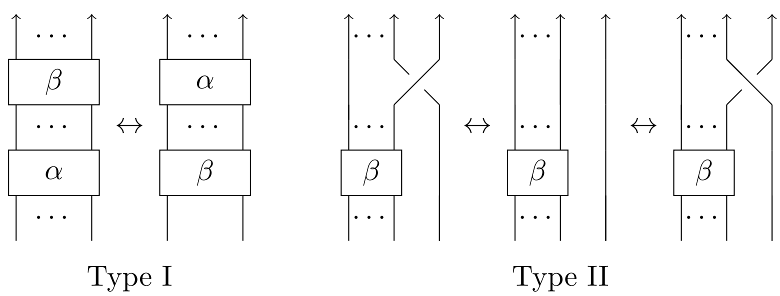


Figure 2.4: Markov Moves

2.3 Hecke Algebras and the HOMFLY-PT Polynomial

Before we discuss the HOMFLY-PT polynomial we will discuss Hecke algebras. The goal of this section is to construct Hecke algebras, and to discuss special properties of these algebras. We will denote the Hecke algebra by $H_n(q,a)$ or simply H_n when q and a are understood. In particular we will discuss a representation of $\mathcal{B}_n$ on H_n.

Definition 2.3.1. The Hecke algebra $H_n(q,a)$ is the $\mathbb{Z}(q,a)$-algebra generated by invertible elements T_i for $i = 1, ..., n-1$ in the following presentation.

$$\begin{aligned} H_n(q,a) = \langle T_1, ..., T_{n-1} |\, & T_iT_j = T_jT_i \text{ for } |i-j| \geq 2, \\ & T_i^2 = a^{-2} + (q - q^{-1})T_i, \\ & T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \rangle. \end{aligned}$$

Note that the relations for H_n look similar to the relations for the braid group $\mathcal{B}_n$. This hints at the possibility of a representation of $\mathcal{B}_n$ on H_n. We will define a representation $\rho : \mathcal{B}_n \to \text{Aut}(H_n)$ by setting $\rho(\sigma_i) = T_i$. Here we use T_i to represent both the element in H_n and the automorphism of H_n which is multiplication by T_i. We now want to define a trace on H_n.

Theorem 2.3.2 (Ocneanu, [14])**.** *Fix $z \in \mathbb{C}$. There exists a linear trace $tr : H_n \to \mathbb{C}(q, a)$ uniquely defined by the following axioms.*

1. $tr(xy) = tr(yx)$
2. $tr(1) = \dfrac{a - a^{-1}}{q - q^{-1}}$
3. $tr(xT_n) = z\,tr(x)$

Where $x, y \in H_n$.

We can normalize the generators T_i so that both possible types of the Markov move of type II affect the trace in the same manner. We will omit the details of this normalization and only give the final result.

Theorem 2.3.3 (HOMFLY-PT polynomial, [14, 34])**.** *Let $\beta \in \mathcal{B}_n$ and let $L = \bar{\beta}$. Then $P(L) = tr(\rho(\beta))$ is a link invariant in $\mathbb{Z}(q, a)$ satisfying the skein relation*

$$aP(L_+) - a^{-1}P(L_-) = (q - q^{-1})P(L_0).$$

Here L_+, L_- and L_0 are oriented links which are the same except in a neighborhood of a point where they are shown in Figure 2.5. We define $P(Unknot) = \dfrac{a - a^{-1}}{q - q^{-1}}$.

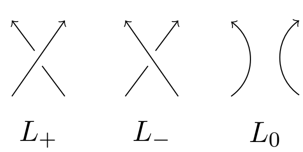


Figure 2.5: Conway Triple

Now we will define the $SL(N)$ polynomial. This polynomial can be obtained by studying the representation theory of $\mathcal{U}_q(\mathfrak{sl}(N))$ (See [31] for example), but we will describe it here in terms of the HOMFLY-PT polynomial.

Definition 2.3.4 ($SL(N)$ polynomial)**.** The $SL(N)$ polynomial is a function $P_n : \{\text{Oriented Links}\} \to \mathbb{Z}(q)$. It is defined by $P_n(L) = P(L)|_{a=q^n}$.

Remark. We can also obtain the Jones and Alexander polynomials from the HOMFLY-PT polynomial as well. The Jones polynomial is the $SL(2)$ polynomial, P_2, and the Alexander polynomial P_0 is found by setting $a = 1$ and $P_0(\text{Unknot}) = 1$.

2.4 MOY Construction of HOMFLY-PT and $SL(N)$ polynomials

Finally we will discuss a construction of the HOMFLY-PT polynomial using polynomial invariants of certain graphs. We will only define what is needed to construct the HOMFLY-PT and $SL(N)$ polynomials. Their construction was first given by Murakami, Ohtsuki and Yamada in 1998 for the $SL(N)$ polynomial in terms of trivalent graphs colored by positive integers [31]. This construction can be extended to the HOMFLY-PT polynomial as well. Our exposition here follows Rasmussen's in [35].

First we define what we will call braid graphs. A *braid graph* is a braid in which all crossings are replaced with singular crossings, which we will mark as the vertex of the graph. An example is shown below in Figure 2.6. The MOY state model of the HOMFLY-PT polynomial resolves a braid into a formal sum of braid graphs with coefficients in $\mathbb{Z}(q, a)$ in the following manner. We may resolve each crossing in two ways; we may remove a crossing entirely and replace it with two vertical arcs, or we may replace a crossing with a singular crossing. To each resolution we assign a weight $\mu \in \mathbb{Z}$. If we resolve the crossing into vertical arcs, we assign the resolution a weight of 0. If we resolve the crossing into a singular crossing, we assign a weight of 1 if the crossing was positive and -1 if the crossing was negative.

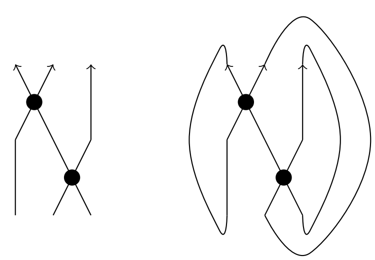


Figure 2.6: A Braid Graph and its Closure

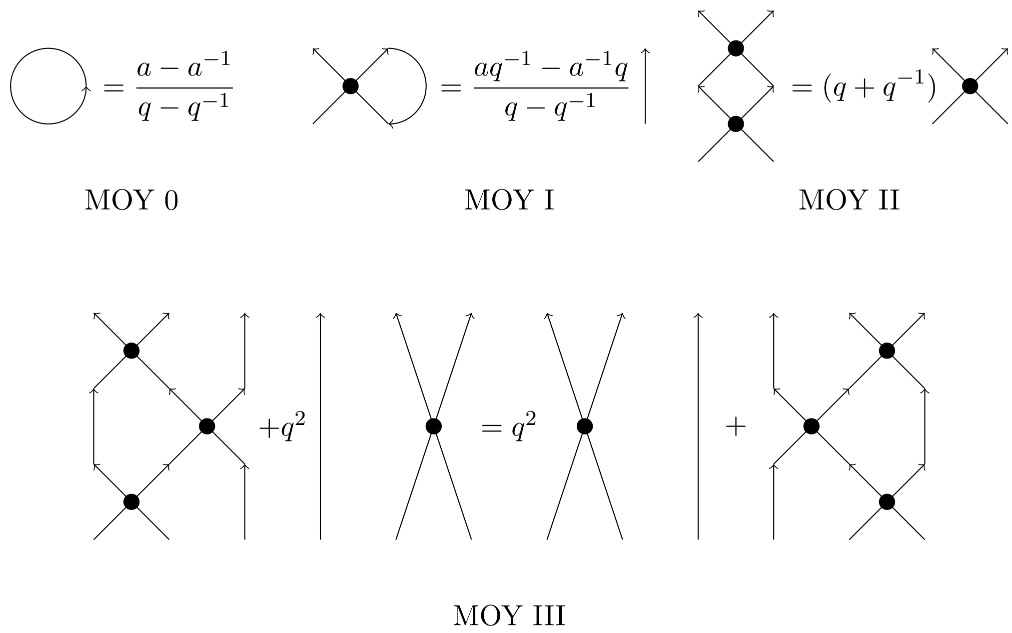


Figure 2.7: MOY Relations for Braid Graphs

Now let D be a link diagram in the form of a braid closure. A *state* of the diagram, σ, is a choice of resolution for each crossing in the link diagram. If a diagram has n crossings then it will have 2^n possible states. We define the weight of a state, $\mu(\sigma)$, to be sum of the weights of each resolution in the state. Also note that each state σ gives rise a closure of a braid graph D_σ. In the paper by Murakami, Ohtsuki, and Yamada [31] it was shown that the unnormalized HOMFLY-PT polynomial can be expressed as a sum

$$P(L) = (aq^{-1})^{w(D)} \sum_{\sigma} (-q)^{\mu(\sigma)} P(D_\sigma).$$

Where $P(D_\sigma)$ is defined by graph isotopy and the relations, which we will refer to as the MOY relations, in Figure 2.7. It can be shown that these relations are sufficient to compute the HOMFLY-PT polynomial.

These braid graphs give a graphical representation of the Kazdhan-Luzstig basis of $H_n(q, a)$. Define $b_i = 1 - q^{-2}T_i$, then the Hecke algebra $H_n(q, a)$ has the following presentation.

$$
\begin{aligned}
H_n(q,a) = \langle b_1, ..., b_{n-1} |\, b_i b_j &= b_j b_i \text{ for } |i-j| \geq 2, \\
b_i^2 &= (1+q^2) b_i, \\
b_i b_{i+1} b_i + q^2 b_{i+1} &= b_{i+1} b_i b_{i+1} + q^2 b_i \rangle.
\end{aligned}
$$

We can associate the braid graphs shown in Figure 2.8 to b_i. If we define multiplication of braid graphs by vertical concatenation as we did with braids, the MOY relations are exactly these relations for the Kazdhan-Lusztig basis.

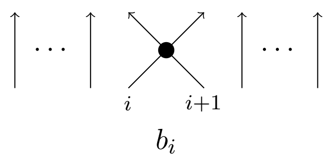


Figure 2.8: Kazdhan-Luzstig Basis represented by Braid Graphs

We may also write a MOY state model for the $SL(N)$ polynomial as well. This is what was originally presented in [31]. In this presentation we only need perform the substitution $a = q^n$ to recieve the $SL(N)$ polynomial.

Remark. It is not necessary for the computation of $P(L)$ and $P_n(L)$ by MOY graphs to take L to be a braid closure. In the original paper by Murakami, Ohtsuki, and Yamada, they compute $P_n(L)$ from an arbitrary link diagram. However in the sequel, it is strictly necessary for us to consider L as a braid closure to compute HOMFLY-PT homology.

CHAPTER 3: HOMFLY-PT Homology

In 2004, Khovanov and Rozansky announced a categorification of the $SL(N)$ polynomial using $\mathbb{Z}_2-$graded matrix factorizations of potentials being sums and differences of x_i^{n+1} [23]. In 2005 they also announced a generalization of this which gave a triply-graded cohomology theory categorifying the HOMFLY-PT polynomial using $\mathbb{Z}_2-$graded matrix factorizations of potentials being sums and differences of $a(x_i - x_j)$ [24]. In 2006, Khovanov made a connection between this theory and Rouquier's braid group action on the objects of a certain homotopy category using Soergel bimodules [21, 38]. In this chapter we will discuss Soergel bimodules, Hochschild homology, and how these can be used to define Khovanov and Rozansky's HOMFLY-PT homology. Our approach will mainly follow [21].

3.1 Soergel Bimodules

Let $R = \mathbb{Q}[x_1, ...x_n]$ and let $R^i = \mathbb{Q}[x_1, ..., x_{i-1}, x_i + x_{i+1}, x_i x_{i+1}, x_{i+2}, ..., x_n]$. The ring R^i is the ring of polynomials which are symmetric in x_i and x_{i+1}, or equivalently the polynomials fixed by the action of the permutation which interchanges i and $i + 1$. It is clear that R is a free R^i-module of rank 2. We will also make the rings R and R^i graded by setting $\deg_q(x_i) = 2$, and call this grading the "q-grading" or the "internal grading". Define the R-bimodule $B_i = R \otimes_{R^i} R$. This is a graded bimodule inheriting the grading from R. B_i is free of rank 2 as both a left R-module and a right R-module.

Definition 3.1.1. Let $R = \mathbb{Q}[x_1, ..., x_n]$. The *category of Soergel bimodules*, $\mathcal{S}$, is the full subcategory of graded R-bimodules generated by tensor products of B_i, their direct summands, and grading shifts.

These bimodules are named after Wolfgang Soergel, who explained their importance in infinite-dimensional representation theory of simple Lie algebras studying Kazhdan-Lusztig theory [39, 40]. In the specific case that we have defined, these bimodules give a categorification of the Kazhdan-

Lusztig basis of $H_n(q,a)$ mentioned in the previous chapter. The following proposition will state exactly what is meant by this statement.

Proposition 3.1.2 ([39])**.** *Let $\mathbf{q}^n$ denote the grading shift functor that raises the q-grading of a bimodule by n. Then there are isomorphisms of graded R-bimodules*

$$B_i \otimes_R B_j \cong B_j \otimes B_i \text{ , if } |i-j| \geq 1,$$

$$B_i \otimes_R B_i \cong B_i \oplus \mathbf{q}^2 B_i,$$

$$(B_i \otimes_R B_{i+1} \otimes_R B_i) \oplus \mathbf{q}^2 B_{i+1} \cong \mathbf{q}^2 B_i \oplus (B_{i+1} \otimes_R B_i \otimes_R B_{i+1}).$$

The first two isomorphisms follow directly from definitions. The third isomorphism is non-trivial. Let $R^{i,i+1}$ be the subring of R fixed by any permutation permuting i, $i+1$ and $i+2$. Now define $B_{i,i+1} = R \otimes_{R^{i,i+1}} R$. Then Soergel in [39] shows there are R-bimodule isomorphisms

$$B_i \otimes_R B_{i+1} \otimes_R B_i \cong B_{i,i+1} \oplus \mathbf{q}^2 B_i,$$

$$B_{i+1} \otimes_R B_i \otimes_R B_{i+1} \cong B_{i,i+1} \oplus \mathbf{q}^2 B_{i+1}.$$

The last isomorphism in the proposition follows immediately. We will call the relations in Proposition 3.1.2 *categorified MOY relations* for reasons which will become clear in what follows. We want to also want to describe the indecomposable Soergel bimodules.

Theorem 3.1.3 (Soergel [41])**.** *For any fully reduced word $w = s_{i_1} \cdots s_{i_k} \in S_n$, there exists up to isomorphism a unique indecomposable Soergel bimodule B_w which occurs as a direct summand of $BS(w) = B_{i_1} \otimes_R \cdots \otimes_R B_{i_k}$, but does not occur as a summand of $BS(w')$ for any $w' < w$ with respect to the Bruhat order. Reduced words $w \in S_n$ give a full description of indecomposable Soergel bimodules up to grading shifts.*

Example 3.1.4. Let we let $n = 3$, then the indecomposable Soergel bimodules are

$$\{R, B_1, B_2, B_{s_1 s_2}, B_{s_2 s_1}, B_{s_1 s_2 s_1}\},$$

where $B_{s_1s_2} = B_1 \otimes_R B_2, B_{s_2s_1} = B_2 \otimes_R B_1$, and $B_{s_1s_2s_1} = B_{i,i+1}$.

We want to now introduce a system of diagrammatics to help us visualize computations with Soergel bimodules. Our diagrammatics will follow almost exactly the conventions of Khovanov and Rozansky [21, 22]. All Soergel bimodules will be represented by an upward oriented graph with each incoming and outgoing strand labelled with a variable. First we will denote $R = \mathbb{Q}[x_1, ..., x_n]$ by a diagram with n vertical lines oriented upward. We will denote B_i using the same diagram that we used for the Kazdhan-Lusztig basis element b_i.

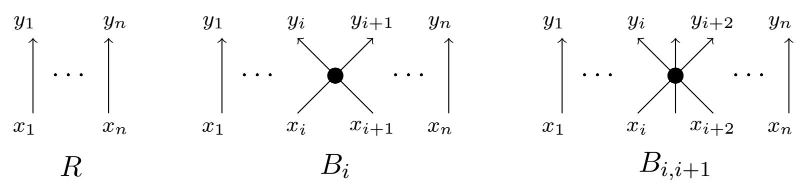


Figure 3.1: Diagrams for R, B_i and $B_{i,i+1}$

We will also denote $B_{i,i+1}$ by the diagram shown in Figure 3.1. If we wish to take the tensor product of Soergel bimodules, say $M \otimes_R N$, we will denote this by taking the diagram for M and gluing it on top of the diagram for N. We will leave marks on the diagram to denote where the gluing occurs. Once we introduce standard bimodules and virtual crossings we will discuss these markings in more detail. Finally we may place diagrams adjacently horizontally as well. If we place the diagrams for M and N in such a manner this will correspond to $M \otimes_{\mathbb{Q}} N$.

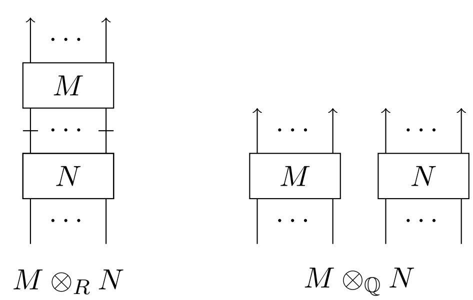


Figure 3.2: Diagrams for $M \otimes_R N$ and $M \otimes_{\mathbb{Q}} N$

3.2 Rouquier's Braid Group Action

Let $R = \mathbb{Q}[x_1, ..., x_n]$. Let $\mathcal{S}$ denote the category of Soergel bimodules. Then let $\mathcal{K}om(\mathcal{S})$ be the homotopy category of Soergel bimodules, that is the category of chain complexes of Soergel bimodules up to homotopy equivalence. In [38], Rouquier proposed a categorification of $\mathcal{B}_n$ using chain complexes of Soergel bimodules in $\mathcal{K}om(\mathcal{S})$. In this section we will describe his construction.

Definition 3.2.1. The category of n-braids, which we will denote by $\mathbb{B}_n$, is the category whose objects are n-braids and whose morphisms are braid cobordisms.

First we will define two morphisms $\chi_+ : R \to B_i$ and $\chi_- : B_i \to R$. R has a single generator, 1, in q-degree 0. Therefore we will define

$$\chi_+(1) = x_i \otimes 1 - 1 \otimes x_{i+1}.$$

Also B_i has a single generator, $1 \otimes 1$, in q-degree 0. We can see this by considering B_i as a quotient of $R \otimes R$, which we will do whenever convenient. We will define

$$\chi_-(1 \otimes 1) = 1.$$

Note that $\deg_q(\chi_+) = 2$ and $\deg_q(\chi_-) = 0$. Now we will define Rouquier's braid group action on $\mathcal{K}om(\mathcal{S})$. Define $\mathcal{F} : \mathbb{B}_n \to \mathcal{K}om(\mathcal{S})$ by

$$\mathcal{F}(1) = \quad 0 \longrightarrow R \longrightarrow 0,$$

$$\mathcal{F}(\sigma_i) = \quad 0 \longrightarrow R \xrightarrow{\chi_+} \mathbf{tq}^{-2} B_i \longrightarrow 0,$$

$$\mathcal{F}(\sigma_i^{-1}) = \quad 0 \longrightarrow \mathbf{t}^{-1} B_i \xrightarrow{\chi_-} R \longrightarrow 0,$$

$$\mathcal{F}(\beta\beta') = \mathcal{F}(\beta) \otimes_R \mathcal{F}(\beta') \text{ for } \beta, \beta' \in \mathbb{B}_n.$$

Here we use a nonstandard notation for homological degree. We will use $\mathbf{t}^k$ to denote that the term sits in homological degree k. Therefore both χ_- and χ_+ have homological degree 1. Also note the q-degree shift of B_i in $\mathcal{F}(\sigma_i)$. We introduce this shift so that the map χ_+ has q-degree 0.

Theorem 3.2.2 (Rouquier [38]). *Let $\mathcal{F}$ be defined as above, then*

1. $\mathcal{F}(\sigma_i\sigma_i^{-1}) \simeq \mathcal{F}(1)$.

2. $\mathcal{F}(\sigma_i\sigma_{i+1}\sigma_i) \simeq \mathcal{F}(\sigma_{i+1}\sigma_i\sigma_{i+1})$.

Here $\simeq$ denotes homotopy equivalence in $\mathcal{K}om(\mathcal{S})$.

We will discuss an outline of the proof of this theorem. When we are trying to prove the analogous statements for the filtered case, the outline of the proof will be similar. First we give a lemma which will help to simplify chain complexes. A proof of this lemma may be found in [4].

Lemma 3.2.1 (Gaussian Elimination). *Consider the complex,*

$$A \xrightarrow{\begin{pmatrix} \bullet \\ \alpha \end{pmatrix}} B \oplus C \xrightarrow{\begin{pmatrix} \varphi & \lambda \\ \mu & \nu \end{pmatrix}} D \oplus E \xrightarrow{\begin{pmatrix} \bullet & \varepsilon \end{pmatrix}} F,$$

in any additive category where $\varphi : B \to D$ is an isomorphism and all other maps are arbitrary up to the condition that $d^2 = 0$. Then there exists a homotopy equivalence

$$\begin{tikzcd}
A \arrow[r, "\begin{pmatrix} \bullet \\ \alpha \end{pmatrix}"] \arrow[d, shift right, "(1)"'] & B \oplus C \arrow[r, "\begin{pmatrix} \varphi & \lambda \\ \mu & \nu \end{pmatrix}"] \arrow[d, shift right, "\begin{pmatrix} 0 & 1 \end{pmatrix}"'] & D \oplus E \arrow[r, "\begin{pmatrix} \bullet & \varepsilon \end{pmatrix}"] \arrow[d, shift right, "\begin{pmatrix} -\mu\varphi^{-1} & 1 \end{pmatrix}"'] & F \arrow[d, shift right, "(1)"'] \\
A \arrow[r, "(\alpha)"] \arrow[u, shift right, "(1)"'] & C \arrow[r, "(\nu - \mu\varphi^{-1}\varepsilon)"] \arrow[u, shift right, "\begin{pmatrix} -\varphi^{-1}\lambda \\ 1 \end{pmatrix}"'] & E \arrow[r, "(\varepsilon)"] \arrow[u, shift right, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"'] & F \arrow[u, shift right, "(1)"']
\end{tikzcd}$$

First we will give an outline of the proof for (1) of Theorem 3.2.2. By definition

$$\mathcal{F}(\sigma_i\sigma_i^{-1}) = \mathcal{F}(\sigma_i) \otimes_R \mathcal{F}(\sigma_i^{-1}),$$

so we can present $\mathcal{F}(\sigma_i\sigma_i^{-1})$ as a complex

$$
\mathcal{F}(\sigma_i\sigma_i^{-1}) \;=\;
\begin{tikzcd}
 & R \arrow[dr, "\chi_+"] & \\
\mathbf{t}^{-1}B_i \arrow[ur, "\chi_-"] \arrow[dr, "\chi_+"'] & \oplus & \mathbf{t}\mathbf{q}^{-2}B_i \\
 & \mathbf{q}^{-2}B_i \otimes_R B_i \arrow[ur, "\chi_-"'] &
\end{tikzcd}
$$

Proposition 3.1.2 tells us that $B_i \otimes_R B_i \cong B_i \oplus \mathbf{q}^2 B_i$. We can apply this isomorphism to $\mathcal{F}(\sigma_i\sigma_i^{-1})$ to get the following complex.

$$
\mathcal{F}(\sigma_i\sigma_i^{-1}) \;\simeq\;
\begin{tikzcd}
 & R \arrow[dr, "\chi_+"] & \\
\mathbf{t}^{-1}B_i \arrow[ur, "\chi_-"] \arrow[dr, "i_2"'] & \oplus & \mathbf{t}\mathbf{q}^{-2}B_i \\
 & \mathbf{q}^{-2}B_i \oplus B_i \arrow[ur, "\pi_1"'] &
\end{tikzcd}
$$

Here i_2 is the inclusion map into the second direct summand and π_1 is the projection map onto the first direct summand. Therefore by Gaussian Elimination we see that

$$
\mathcal{F}(\sigma_i\sigma_i^{-1}) \simeq (0 \longrightarrow R \longrightarrow 0) \;= \mathcal{F}(1).
$$

Now we will give an outline of the proof for (2) in Theorem 3.2.2. The basic idea of this proof is to show that $\mathcal{F}(\sigma_i\sigma_{i+1}\sigma_i)$ and $\mathcal{F}(\sigma_{i+1}\sigma_i\sigma_{i+1})$ are both homotopy equivalent to a complex which is fixed by the permutation switching x_1 and x_3. The permutation corresponds to reflecting a braid across a vertical axis. We can write $\mathcal{F}(\sigma_i\sigma_{i+1}\sigma_i)$ as a cube. We will abbreviate the terms in the complex below by writing MN for $M \otimes_R N$.

$$
\mathcal{F}(\sigma_i\sigma_{i+1}\sigma_i)=
\begin{tikzcd}
R \arrow[rr] \arrow[dr] \arrow[dd] & & \mathbf{t}\mathbf{q}^{-2}B_i \arrow[dr] \arrow[dd] & \\
& \mathbf{t}\mathbf{q}^{-2}B_i \arrow[rr] \arrow[dd] & & \mathbf{t}^2\mathbf{q}^{-4}B_iB_i \arrow[dd] \\
\mathbf{t}\mathbf{q}^{-2}B_{i+1} \arrow[rr] \arrow[dr] & & \mathbf{t}^2\mathbf{q}^{-4}B_{i+1}B_i \arrow[dr] & \\
& \mathbf{t}^2\mathbf{q}^{-4}B_iB_{i+1} \arrow[rr] & & \mathbf{t}^3\mathbf{q}^{-6}B_iB_{i+1}B_i
\end{tikzcd}
$$

In the above complex, the maps are omitted. By Proposition 3.1.2, we know that $B_i \otimes_R B_i \cong B_i \oplus \mathbf{q}^2 B_i$ and $B_i \otimes_R B_{i+1} \otimes_R B_i \cong B_{i,i+1} \oplus \mathbf{q}^2 B_i$. Therefore by applying these isomorphisms to $\mathcal{F}(\sigma_i\sigma_{i+1}\sigma_i)$ we get a new complex.

$$
\mathcal{F}(\sigma_i\sigma_{i+1}\sigma_i)\simeq
\begin{tikzcd}
R \arrow[rr] \arrow[dr] \arrow[dd] & & \mathbf{t}\mathbf{q}^{-2}B_i \arrow[dr, "\left(\begin{smallmatrix} * & 1 \end{smallmatrix}\right)"] \arrow[dd] & \\
& \mathbf{t}\mathbf{q}^{-2}B_i \arrow[rr] \arrow[dd] & & \mathbf{t}^2\mathbf{q}^{-4}B_i \oplus \mathbf{t}^2\mathbf{q}^{-2}B_i \arrow[dd, "\left(\begin{smallmatrix} * & * \\ 1 & * \end{smallmatrix}\right)"] \\
\mathbf{t}\mathbf{q}^{-2}B_{i+1} \arrow[rr] \arrow[dr] & & \mathbf{t}^2\mathbf{q}^{-4}B_{i+1}B_i \arrow[dr] & \\
& \mathbf{t}^2\mathbf{q}^{-4}B_iB_{i+1} \arrow[rr] & & \mathbf{t}^3\mathbf{q}^{-6}B_{i,i+1} \oplus \mathbf{t}^3\mathbf{q}^{-4}B_i
\end{tikzcd}
$$

We have placed *'s in the place of maps which are irrelevant to this discussion. Therefore we may apply Gaussian Elimination to this complex to get the below complex.

$$\begin{tikzcd}
 & & \mathbf{t}\mathbf{q}^{-2}B_i \arrow[r] \arrow[ddr] & \mathbf{t}^2\mathbf{q}^{-4}B_iB_{i+1} \arrow[dr] & \\
\mathcal{F}(\sigma_i\sigma_{i+1}\sigma_i) \simeq & R \arrow[ur] \arrow[dr] & & & \mathbf{t}^3\mathbf{q}^{-6}B_{i,i+1} \\
 & & \mathbf{t}\mathbf{q}^{-2}B_{i+1} \arrow[r] \arrow[uur] & \mathbf{t}^2\mathbf{q}^{-4}B_{i+1}B_i \arrow[ur] &
\end{tikzcd}$$

The above complex can be shown to be "symmetric", that is the complex is fixed under the permutation of x_1 and x_3. It can be easily shown that $\mathcal{F}(\sigma_{i+1}\sigma_i\sigma_{i+1})$ is homotopy equivalent to the same complex but with x_3 and x_1 permuted. This completes the outline of the proof of Theorem 3.2.2.

Remark. $\mathcal{F} : \mathbb{B}_n \to \mathcal{K}om(\mathcal{S})$ is actually a functor. It was shown by Elias and Krasner that this construction was functorial with respect to braid cobordisms [12]. More precisely, if S and S' are equivalent braid cobordisms from β to β', then $\mathcal{F}(S)$ and $\mathcal{F}(S')$ are homotopy equivalent maps from $\mathcal{F}(\beta)$ to $\mathcal{F}(\beta')$.

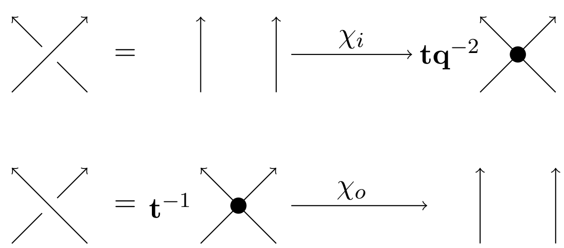


Figure 3.3: Diagrammatic form of $\mathcal{F}(\sigma_1)$ and $\mathcal{F}(\sigma_1^{-1})$

We end this section by giving a diagrammatic form for $\mathcal{F}(\sigma_i)$ and $\mathcal{F}(\sigma_i^{-1})$. To do so, we simply replace the terms in the complex with their respective diagrams. We show this in Figure 3.3 in the case that $n = 2$.

3.3 Koszul Resolutions

In this section we will define Koszul complexes and describe when these complexes are free resolutions of R-modules, and then we will use this to define the Hochschild homology of Soergel bimodules. Finally we will introduce diagrammatics for Hochschild homology and explain our choice of diagrammatics. Our exposition here follows [21, 22, 45].

Suppose for this discussion that R is a commutative ring with unity. The ring $R = \mathbb{Q}[x_1, ..., x_n]$ will be the primary example we will keep in mind. Let $p \in R$, then let $K(p)$ be the chain complex

$$K(p) = \quad 0 \longrightarrow R \xrightarrow{\ p\ } R \longrightarrow 0$$

with terms concentrated in homological degree 0 and 1. If R is a graded ring with $\deg_q(p) = k$, then we shift the copy of R is homological degree 1 by $\mathbf{q}^k$ to make multiplication by p a degree 0 map. That is,

$$K(p) = \quad 0 \longrightarrow \mathbf{q}^k R \xrightarrow{\ p\ } R \longrightarrow 0.$$

Definition 3.3.1. Let $\mathbf{p} = (p_1, ..., p_n)$ be a sequence of elements in R. We define the *Koszul complex*, $K(\mathbf{p})$, as the tensor product complex

$$K(\mathbf{p}) = K(p_1) \otimes_R \cdots \otimes_R K(p_n).$$

Proposition 3.3.2. *Suppose* $\mathbf{p}$ *is a regular sequence in* R*. That is,* p_i *is not a zero divisor in* $R/(p_1, ..., p_{i-1})R$ *for all* i*. Then*

$$H_k(K(\mathbf{p})) = \begin{cases} 0 & \text{if } k \neq 0 \\ R/\mathbf{p}R & \text{if } k = 0 \end{cases}$$

A proof of this proposition can be found in Weibel [45].

Corollary 3.3.3. *Let* $\mathbf{p}$ *be a regular sequence in* R*. Then* $K(\mathbf{p})$ *is a free resolution of* R*. In* *particular if A is an R-module then,*

$$Tor_k^R(R/\mathbf{p}R, A) = H_k(K(\mathbf{p}) \otimes_R A)$$
$$Ext_R^k(R/\mathbf{p}R, A) = H^k(Hom(K(\mathbf{p}), A))$$

Before moving on we want to introduce an alternate notation for Koszul complexes. Suppose $\mathbf{p} = (p_1, ..., p_n)$ is a sequence of elements in R, then will also present $K(\mathbf{p})$ as a column vector,

$$K(\mathbf{p}) = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$$

From this point forward we will use this presentation whenever it convenient. We will also use brackets instead of parentheses for the vectors to minimize possible confusion. An advantage of this presentation is that we can see change of basis transformations as row operations.

Proposition 3.3.4. *Let $\mathbf{p} = (p_1, ..., p_n)$ be a sequence in a R. Then there exists an invertible chain map $\Phi^k_{i\to j}$ such that*

$$\Phi^k_{i\to j} : \begin{bmatrix} p_i \\ p_j \end{bmatrix} \to \begin{bmatrix} p_i \\ kp_i + p_j \end{bmatrix}$$

and all other rows are fixed.

Proof. The map $\Phi^k_{i\to j}$ is defined as

$$\begin{tikzcd} R \arrow[r, "\binom{p_i}{p_j}"] \arrow[d, "1"'] & R \oplus R \arrow[r, "\begin{pmatrix} -p_j & p_i \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}"'] & R \arrow[d, "1"'] \\ R \arrow[r, "\binom{p_i}{kp_i+p_j}"'] & R \oplus R \arrow[r, "\begin{pmatrix} -kp_i-p_j & p_i \end{pmatrix}"'] & R \end{tikzcd}$$

On the tensor factors corresponding to p_i and p_j and is the identity on all other factors. This map is clearly invertible. □

3.4 Hochschild Homology of Soergel Bimodules

We will begin by defining Hochschild homology, and then look at the Hochschild homology of Soergel bimodules. Our definitions and basic results about Hochschild homology follow [45]. Hochschild homology is a homology theory used to study properties of algebras and their bimodules.

Let k be a field, let R be a k-algebra, and set $R^e = R \otimes_k R$. Note that if M is a R^e-module, then it is also a R-bimodule. The basic example we will keep in mind is $k = \mathbb{Q}$ and $R = \mathbb{Q}[x_1, ..., x_n]$.

Definition 3.4.1. Let M be a R-bimodule, the we define the *Hochschild homology* of R with coefficients in M to be

$$\mathrm{HH}_k(M, R) = \mathrm{Tor}_k^{R^e}(M, R).$$

Also we define

$$\mathrm{HH}(M, R) = \bigoplus_{k \geq 0} \mathrm{HH}_k(M, R).$$

If R is understood, then we may abbreviate $\mathrm{HH}(M) = \mathrm{HH}(M, R)$.

Remark. We may also define *Hochschild cohomology* by setting

$$\mathrm{HH}^k(M, R) = \mathrm{Ext}_{R^e}^k(M, R).$$

However we will only be using Hochschild homology in the sequel.

Proposition 3.4.2. *Let $k = \mathbb{Q}$ and let $R = \mathbb{Q}[x_1, ..., x_n]$. Then*

$$HH_k(R, R) \cong \Lambda^k(R^n).$$

Proof. First we want to carefully describe R as an R-bimodule. Recall that any R-bimodule can also be thought of as an $R \otimes_{\mathbb{Q}} R$-module. From basic commutative algebra we know that $R \otimes_{\mathbb{Q}} R \cong \mathbb{Q}[x_1, ..., x_n, y_1, ..., y_n]$, so then clearly $R \cong (R \otimes_{\mathbb{Q}} R)/I$ where $I = (x_1 - y_1, ..., x_n - y_n)$. The sequence $\mathbf{p} = (x_1 - y_1, ...x_n - y_n)$ can easily be shown to be a regular sequence. Therefore by Corollary 3.3.3, $K(\mathbf{p})$ is a free R^e-module resolution of R. That is,

$$K(\mathbf{p}) = \bigotimes_{i=1}^{k} (R^e \xrightarrow{x_i - y_i} R^e).$$

Therefore the functor $\bullet \otimes_{R^e} R$ identifies the variables x_i and y_i so that

$$K(\mathbf{p}) \otimes_{R^e} R = \bigotimes_{i=1}^{k} (R \xrightarrow{0} R).$$

From this we see that $\mathrm{HH}_k(M, R) = \mathrm{Tor}_k^{R^e}(R, R) = R^{\binom{n}{k}} \cong \Lambda^k(R^n)$. $\square$

Finally we will mention some useful basic and "trace-like" properties of Hochschild homology.

Theorem 3.4.3. *Let k be a field, R an k-algebra and M, N be R^e-modules. The following properties hold for Hochschild homology.*

1. *$HH_k(\bullet, R)$ and $HH(\bullet, R)$ are additive functors.*

2. *$HH_k(M \oplus N, R) \cong HH_k(M, R) \oplus HH_k(N, R)$.*

3. *$HH_k(M \otimes_R N, R) \cong HH_k(N \otimes_R M, R)$.*

4. *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of R^e-modules, then there exists a long exact sequence*

$$\cdots \xrightarrow{\partial} HH_k(M_1, R) \to HH_k(M_2, R) \to HH_k(M_3, R) \xrightarrow{\partial} HH_{k-1}(M_1, R) \to \cdots$$

Proof. (1) and (2) follow from the properties of the $\mathrm{Tor}_k^{R^e}(\bullet, R)$ functor. Property (3) follows from looking at projective resolutions of $M \otimes_R N$ and of $N \otimes_R M$. Finally (4) is an exercise in [45]. $\square$

Now we return to discussing Soergel bimodules. We want to be able to compute the Hochschild homology $\mathrm{HH}(M, R)$ for any Soergel bimodule M. To do so we need to know more about the general structure of these bimodules.

Proposition 3.4.4. *Let M be a Soergel bimodule. Then as a R^e-module,*

$$M \cong R^e/I_1 \oplus \cdots \oplus R^e/I_m.$$

Furthermore, each I_i is generated by a regular sequence in R^e.

Proof. By Theorem 3.1.3, any Soergel bimodule M can be expressed as a sum of indecomposable Soergel bimodules,

$$M \cong \bigoplus_{i=1}^{n} \mathbf{q}^{k_i} B_{w_i},$$

where $w_i \in S_n$. In [21], Khovanov shows that every indecomposable Soergel bimodule can be written in the form R^e/I where I is an ideal generated by a regular sequence. $\square$

By Proposition 3.4.4 we know that the Koszul complex for any Soergel bimodule is a free R^e-module resolution of M. Therefore we can use this to compute the Hochschild homology. Here we introduce a grading convention for Hochschild homology. We will use the notation $\mathbf{a}^k p$ to denote that the Hochschild degree or a-degree, $\deg_a(p)$, of p is being shifted up by k. In particular if $p \in \mathrm{HH}_j(M, R)$ then $\mathbf{a}^k p \in \mathrm{HH}_{j+k}(\mathbf{a}^k M, R)$. By convention we will say that $\deg_a(p) = 0$ for all $p \in R$. We will also use the same notation $\mathbf{a}^k$ to denote that a term lies in homological degree k in a Koszul complex. This notation is consistent as these terms become Hochschild chains after applying the left derived tensor product.

Example 3.4.5. Let $R = \mathbb{Q}[x_1, x_2]$ and $M = B_1 = R \otimes_{R^1} R$. By studying the left and right actions by R on B_1 we see that $B_1 \cong \mathbb{Q}[x_1, x_2, y_1, y_2]/(x_1 + x_2 - y_1 - y_2, x_1x_2 - y_1y_2)$ as R^e-modules. $\mathbf{p} = (x_1 + x_2 - y_1 - y_2, x_1x_2 - y_1y_2)$ is a regular sequence, so then the Koszul complex $K(\mathbf{p})$ is a free resolution. Recall that Soergel bimodules are also graded with $\deg_q(x_i) = 2$. The same will be true for y_i as it represents the right action by x_i. Therefore,

$$K(\mathbf{p}) = (\mathbf{aq}^4 R^e \xrightarrow{x_1x_2 - y_1y_2} R^e) \otimes_{R^e} (\mathbf{aq}^2 R^e \xrightarrow{x_1 + x_2 - y_1 - y_2} R^e).$$

Once again, the functor $\bullet \otimes_{R^e} R$ identifies the variables x_i and y_i so that

$$K(\mathbf{p}) = (\mathbf{aq}^4 R \xrightarrow{0} R) \otimes_R (\mathbf{aq}^2 R \xrightarrow{0} R).$$

This implies that $\mathrm{HH}(B_1, R) = \mathbf{a}^2\mathbf{q}^6 R \oplus \mathbf{aq}^4 R \oplus \mathbf{aq}^2 R \oplus R$. We can also revisit the calculation for $\mathrm{HH}(R, R)$, while keeping track of the gradings, to show that

$$\mathrm{HH}(R, R) = \mathbf{a}^2\mathbf{q}^4 R \oplus \mathbf{aq}^2 R \oplus \mathbf{aq}^2 R \oplus R.$$

Therefore, $\mathrm{HH}(R, R)$ and $\mathrm{HH}(B_1, R)$ are *not* isomorphic as graded objects though they are isomorphic as non-graded objects.

We now introduce some basic diagrammatics for Hochschild homology to coincide with what has already been introduced for Soergel bimodules. Set $R = \mathbb{Q}[x_1, ..., x_n]$, and let M be an R-bimodule. The we will denote $\mathrm{HH}(M, R)$ by closing the strands of the graph as if we were closing a braid. This is shown in Figure 3.4. This diagram hints at the fact that Hochschild homology may be right analog to taking the trace for computing the HOMFLY-PT polynomial.

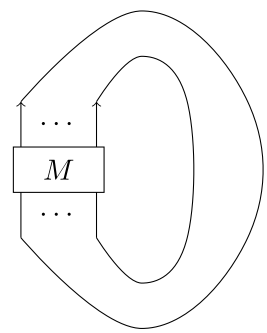


Figure 3.4: Diagram for $\mathrm{HH}(M, R)$

Also recall that from Theorem 3.4.3, we know that $\mathrm{HH}_k(M \otimes_R N, R) \cong \mathrm{HH}_k(N \otimes_R M, R)$. We can visualize this isomorphism by sliding the box for M around the closed strands and moving it to underneath the box for N. This process is shown in Figure 3.5.

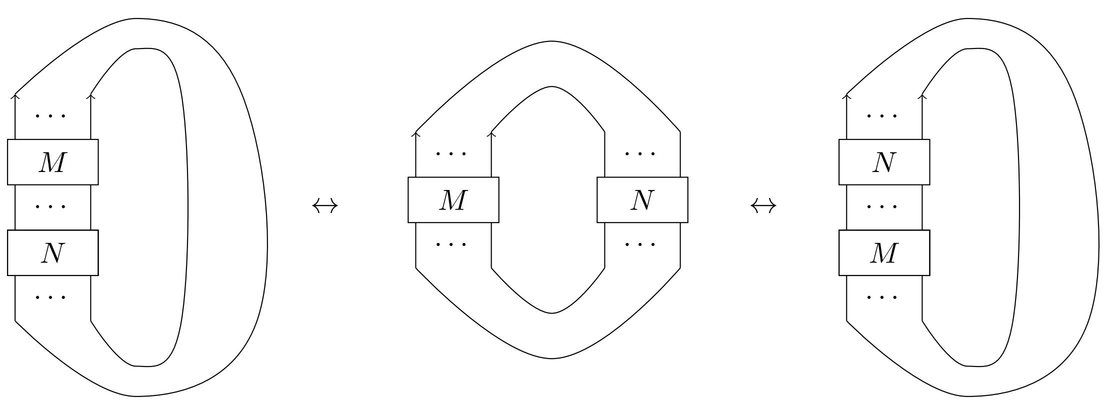


Figure 3.5: Diagram for the isomorphism $\mathrm{HH}_k(M \otimes_R N, R) \cong \mathrm{HH}_k(N \otimes_R M, R)$

3.5 HOMFLY-PT Homology

Before defining HOMFLY-PT homology of a link L, we will introduce shifts to the Rouquier complexes and the Hochschild homology functor. When we are going through the proof of the Markov moves it will become apparent why these shifts are needed. The idea of using fractional shifts was first introduced by Hao Wu in [47]. We will follow that approach here.

Let $\beta \in \mathbb{B}_n$ be a braid representative for a link L. Let $\mathbf{s} = \mathbf{t}^{-1/2}\mathbf{q}\mathbf{a}^{1/2}$. We define the shifted Rouquier complex by $\widetilde{\mathcal{F}}(\sigma_i) = \mathbf{s}\mathcal{F}(\sigma_i)$, and $\widetilde{\mathcal{F}}(\sigma_i^{-1}) = \mathbf{s}^{-1}\mathcal{F}(\sigma_i^{-1})$. Direct computation shows that these shifts do not violate the isomorphisms proved in Theorem 3.2.2. Also as before we set $\widetilde{\mathcal{F}}(\beta\beta') = \widetilde{\mathcal{F}}(\beta) \otimes_R \widetilde{\mathcal{F}}(\beta')$ for $\beta, \beta' \in \mathbb{B}_n$. We also define a shifted Hochschild homology functor $\widetilde{\mathrm{HH}}(\bullet, R) = \mathbf{s}^n\mathbf{a}^{-n}\mathrm{HH}(\bullet, R)$.

Now we describe how to compute $\mathcal{H}(L)$. We first apply the functor $\widetilde{\mathcal{F}}$ to a braid representative of L, β, to receive a chain complex

$$\cdots \xrightarrow{d} \widetilde{\mathcal{F}}_{j-1}(\beta) \xrightarrow{d} \widetilde{\mathcal{F}}_j(\beta) \xrightarrow{d} \cdots.$$

We then apply our shifted Hochschild homology functor $\widetilde{\mathrm{HH}}(\bullet, R)$ to $\widetilde{\mathcal{F}}(\beta)$ to get a new chain complex,

$$\cdots \xrightarrow{\widetilde{\mathrm{HH}}(d)} \widetilde{\mathrm{HH}}(\widetilde{\mathcal{F}}_{j-1}(\beta), R) \xrightarrow{\widetilde{\mathrm{HH}}(d)} \widetilde{\mathrm{HH}}(\widetilde{\mathcal{F}}_j(\beta), R) \xrightarrow{\widetilde{\mathrm{HH}}(d)} \cdots.$$

Define $\mathcal{H}(\beta) = H(\widetilde{\mathrm{HH}}(\widetilde{\mathcal{F}}(\beta), R))$.

Theorem 3.5.1. *(Khovanov, Rozansky [24]) Let L be a link and $\beta \in \mathbb{B}_n$ be a braid representative for L. Then $\mathcal{H}(L) = \mathcal{H}(\beta)$ is a triply-graded vector space which is a link invariant categorifying the HOMFLY-PT polynomial $P(L)$.*

Before briefly discussing a proof of Theorem 3.5.1, we will clarify what is meant by $\mathcal{H}(L)$

categorifying the HOMFLY-PT polynomial $P(L)$. We can express $\mathcal{H}(L)$ as

$$\mathcal{H}(L) = \bigoplus_{j\in\mathbb{Z}, i,k\in\frac{1}{2}\mathbb{Z}} \mathbf{t}^i\mathbf{q}^j\mathbf{a}^k\mathbb{Q}^{d(i,j,k)}.$$

Then by Theorem 3.5.1,

$$P(L) = \sum_{j\in\mathbb{Z}, i,k\in\frac{1}{2}\mathbb{Z}} d(i,j,k)(-1)^{i+k}q^j a^k.$$

We have to do a change of variables to get back to our original conventions for the HOMFLY-PT polynomial. The conventions used for the gradings for $\mathcal{H}(L)$ give

$$P(\text{Unknot}) = \frac{1+aq^2}{1-q^2}.$$

If we let $\alpha = \sqrt{-aq^2}$, and multiply by qa^{-1} then we have

$$P(\text{Unknot}) = \frac{a-a^{-1}}{q-q^{-1}}$$

as in our original conventions.

We know from Theorem 3.2.2 that $\mathcal{H}(L)$ is preserved by Reidemeister II and III. It remains to show that the Markov moves hold as well. the Markov move of type I holds by part (3) of Theorem 3.4.3. It remains to check Markov moves of type II. We will only discuss the proof of the case of positive stabilization, that is $\beta\sigma_n$ being replaced by β.

Define the *closing functor* $\mathcal{C}\ell_i(\bullet) = \bullet \overset{\mathrm{L}}{\otimes}_{\mathbb{Q}[x_i,y_i]} \mathbb{Q}[x_i,y_i]$. We can think of this as a "partial" left derived tensor product, which corresponds to closing only the ith strand of our braid diagram. To prove positive stabilization it will suffice to prove that $\widetilde{\mathcal{C}\ell}_2(\widetilde{\mathcal{F}}(\sigma_1)) \cong \widetilde{\mathcal{F}}(1)$, where $\widetilde{\mathcal{C}\ell}_i(\bullet) = \mathbf{t}^{-1/2}\mathbf{q}\mathbf{a}^{-1/2}\mathcal{C}\ell_i(\bullet)$ is the shifted closing functor and 1 denotes the trivial braid in one strand. In our diagrammatics this corresponds to the positive version of Reidemeister I (See Figure 2.1).

We first consider the Koszul resolutions of R and B_1

$$R = \begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 \end{bmatrix} \quad B_1 = \begin{bmatrix} x_1 + x_2 - y_1 - y_2 \\ x_1x_2 - y_1y_2 \end{bmatrix}.$$

By applying $\Phi^1_{2\to 1}$ to the Koszul complex for R and replacing y_1 with $x_1 + x_2 - y_2$ we may rewrite the Koszul complexes as

$$R = \begin{bmatrix} x_1 + x_2 - y_1 - y_2 \\ x_2 - y_2 \end{bmatrix} \quad B_1 = \begin{bmatrix} x_1 + x_2 - y_1 - y_2 \\ (x_2 - y_2)(x_1 - y_2) \end{bmatrix}.$$

Set $a = x_2 - y_2, b = x_1 - y_2$, and $w = x_1 + x_2 - y_1 - y_2$. Khovanov and Rozansky prove in [24] that we may now write $\mathcal{F}(\sigma_i)$ as the complex

$$\begin{bmatrix} w \\ a \end{bmatrix} \xrightarrow{Id \otimes \psi} \mathbf{tq}^{-2} \begin{bmatrix} w \\ ab \end{bmatrix},$$

where $\psi : K(a) \to K(ab)$ is the map

$$\begin{tikzcd} K(a) = \mathbf{aq}^2 R \arrow[r, "a"] \arrow[d, "1"'] & R \arrow[d, "b"] \\ K(ab) = \mathbf{taq}^2 R \arrow[r, "ab"] & \mathbf{tq}^{-2} R \end{tikzcd}$$

Now we apply $\mathcal{C}\ell_2$ to this complex, this identifies x_2 and y_2. Let $R' = \mathbb{Q}[x_1, x_2, y_1]$, then $\mathcal{C}\ell_2(\mathcal{F}(\sigma_1))$ is

$$\begin{tikzcd} K(a) = \mathbf{aq}^2 R' \arrow[r, "0"] \arrow[d, "1"'] & R' \arrow[d, "x_1 - x_2"] \\ K(ab) = \mathbf{taq}^2 R' \arrow[r, "0"] & \mathbf{tq}^{-2} R'. \end{tikzcd}$$

This complex splits into two direct summands. The first summand,

$$\mathbf{aq}^2 R' \xrightarrow{1} \mathbf{taq}^2 R',$$

is contractible by Gaussian elimination. So we are left with a short complex

$$R' \xrightarrow{x_1 - x_2} \mathbf{tq}^{-2} R'.$$

If we let $R'' = \mathbb{Q}[x_1, y_1]$, we can rewrite this complex as

$$R' \xrightarrow{\begin{pmatrix} 0 \\ x_1 - x_2 \end{pmatrix}} \mathbf{tq}^{-2} R'' \oplus \mathbf{tq}^{-2}(x_1 - x_2) R''.$$

By Gaussian elimination this is homotopy equivalent to $\mathbf{tq}^{-2} R''$. If we instead apply the shifted functors, then $\widetilde{\mathcal{C}\ell}_2(\widetilde{\mathcal{F}}(\sigma_1)) \cong R''$ as desired.

CHAPTER 4: Virtual Crossings and Filtrations

Virtual crossings were first considered as a tool in link homology by Khovanov and Rozansky in [22]; they used virtual crossings to construct a conjectural link homology theory categorifying the Kauffman $SO(2n)$ polynomial. In an appendix of this paper, they also discussed how the concept of virtual crossings could be introduced to HOMFLY-PT homology as well. Also Thiel showed that Rouquier's braid group action on the category of Soergel bimodules could be extended to a virtual braid group action on a slightly larger category of bimodules which includes not only indecomposable Soergel bimodules, but also their irreducible submodules [42].

In this chapter we shall introduce the basics of virtual braids and links. Then we will define what are known as "standard bimodules" and explain how they relate to Soergel bimodules and virtual crossings. We will then show how Soergel bimodules can be filtered by standard bimodules and how these filtrations can be expressed in terms of mapping cones in the proper category.

4.1 Virtual Braids and Links

Virtual links were first introduced by Kauffman in 1996 [18]. They were first defined as equivalence classes of Gauss codes. Gauss codes encode the crossing data of links, up to certain relations which generalize the Reidemeister moves in this setting. We will discuss virtual links from the viewpoint of virtual link diagrams and also speak about the topological interpretation of virtual links as given by Kuperberg [27].

Definition 4.1.1. A *virtual link diagram* is an oriented planar 4-regular graph endowed with the following structure: each vertex is either considered an over crossing, an under crossing, or is marked by what we will call a *virtual crossing*. If a crossing is not virtual then we call it *classical* and if a virtual link diagram has no virtual crossing we call it a *classical link diagram.*

Figure 4.1 shows the vertex corresponding to a virtual crossing and an example of a virtual link diagram for the virtual trefoil knot.

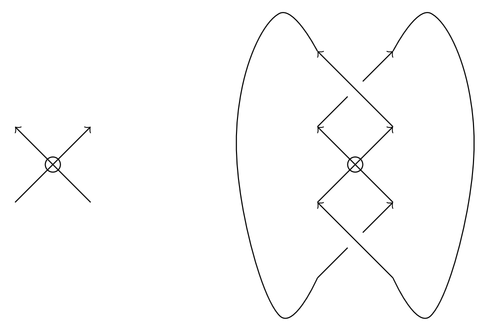


Figure 4.1: Virtual crossing and virtual trefoil knot

Definition 4.1.2. We call two virtual link diagrams *equivalent* if there exists a sequence of generalized Reidemeister moves that transforms one diagram into the other. The generalized Reidemeister moves are the following (see Figure 4.2):

1. Classical Reidemeister moves, that is the moves of Figure 2.1.
2. Virtual versions of the classical Reidemeister moves. We will call these virtual Reidemeister I, II, and III, or briefly VRI, VRII, and VRIII.
3. The semivirtual Reidemeister move, or briefly SVR.

It should be noted that the move in Figure 4.3, called the *forbidden move*, is not allowed.

Definition 4.1.3. A *virtual link* is an equivalence class of virtual link diagrams modulo the generalized Reidemeister moves.

Proposition 4.1.4 ([19]). *If two classical links are related by generalized Reidemeister moves, then they are isotopic in the classical sense. Also if a link only has virtual crossings then it is equivalent to an unlink.*

Kuperberg proposed a topological interpretation of virtual links in 2003 involving embeddings of links on genus g surfaces, S_g, which have been slightly thickened, which we will denote by $\Sigma_g = S_g \times (-\varepsilon, \varepsilon)$ [27]. When we project onto $\mathbb{R}^2 \times (-\varepsilon, \varepsilon)$, actual crossings in Σ_g induce classical crossings in the projection, but crossings induced by the projection between strands which lie in different handles of Σ_g induce virtual crossings in the projection.

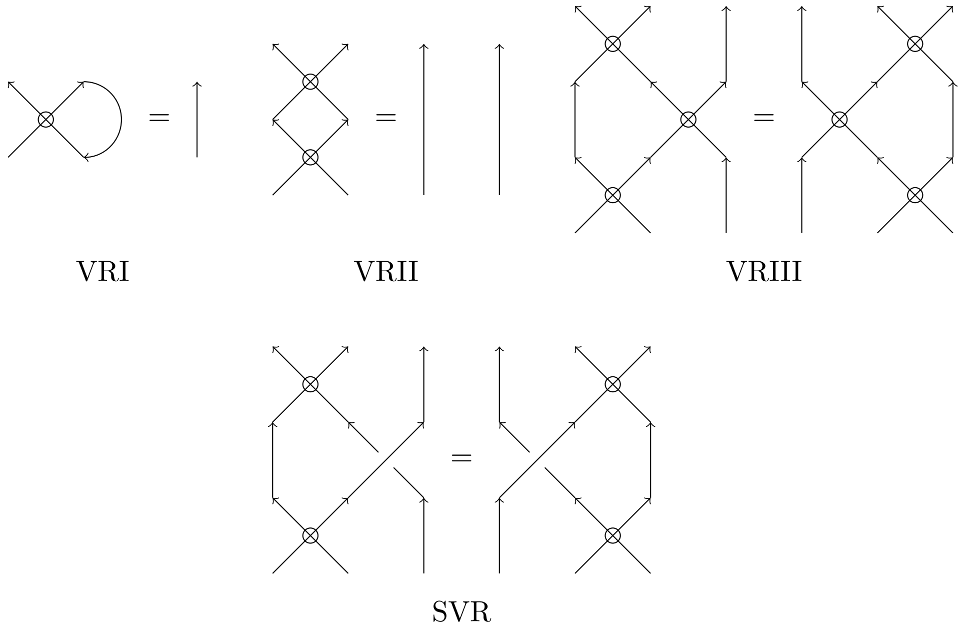


Figure 4.2: Generalized Reidemeister Moves

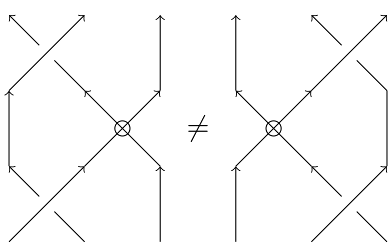


Figure 4.3: Forbidden Move

Let $g(L)$ denote the number of virtual crossings of a virtual link diagram L. Then we have an embedding $\psi : s(L) \to \Sigma_{g(L)}$, where $s(L)$ is a disjoint union of circles equal to the number of components of L.

Definition 4.1.5. We say that two such embeddings are *stably equivalent* if one can be obtained from the other from isotopy in the thickened surfaces, homeomorphisms of surfaces, or the addition or subtraction of handles which are not incident to the images of the curves.

Theorem 4.1.6 (Kuperberg [27])**.** *Two virtual link diagrams generate equivalent virtual links if and only if their corresponding surface embeddings are stably equivalent.*

Now we discuss the virtual braid group and how it relates to virtual links.

Definition 4.1.7. The virtual n-braid group, which we denote by $\mathcal{VB}_n$, is the group generated by the elementary n-braids σ_i and the elementary virtual braids v_i, with the following presentation:

$$
\begin{aligned}
\mathcal{VB}_n = \langle \sigma_1, ..., \sigma_{n-1}, v_1, ..., v_{n-1} | \sigma_i\sigma_j &= \sigma_j\sigma_i \text{ if } |i-j| \geq 2, \\
\sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} \\
v_iv_j &= v_jv_i \text{ if } |i-j| \geq 2, \\
v_iv_{i+1}v_i &= v_{i+1}v_iv_{i+1} \\
v_i^2 &= 1 \\
v_i\sigma_{i+1}v_i &= v_{i+1}\sigma_iv_{i+1} \\
\sigma_iv_j &= v_j\sigma_i \text{ if } |i-j| \geq 2\rangle
\end{aligned}
$$

The first two relations are the classical braiding relations for $\mathcal{B}_n$. The next three relations describe how virtual crossings interact. In particular the relation $v_iv_{i+1}v_i = v_{i+1}v_iv_{i+1}$ represents virtual Reidemeister III and $v_i^2 = 1$ represents virtual Reidemeister II. The last two relations represent how classical and virtual braids interact, in particular the relation $v_i\sigma_{i+1}v_i = v_{i+1}\sigma_iv_{i+1}$ represents the semivirtual Reidemeister move.

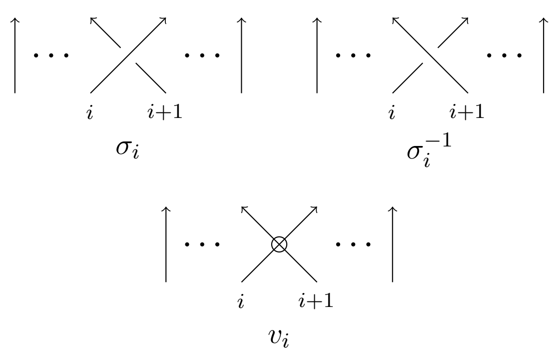


Figure 4.4: Elementary virtual braids

Remark 4.1.1. *Inside of the virtual braid group $\mathcal{VB}_n$ we have two important subgroups. The subgroup generated by the elements v_i is isomorphic to the symmetric group S_n and the subgroup generated by the elements σ_i is the classical braid group $\mathcal{B}_n$.*

As with classical links, there is a deep connection between virtual braids and virtual links. Recall Alexander's Theorem stated that every link is the closure of a braid. A similar statement holds true for virtual links. Let $\beta \in \mathcal{VB}_n$. We define the closure of a virtual braid, $\bar{\beta}$, as we did before with the closure of a braid.

Theorem 4.1.8 (Kauffman [18])**.** *Let L be a virtual link. Then there exists a virtual braid β such that $\bar{\beta}$ is equivalent to L.*

Also similar to the case for classical links, we can tell when two virtual braids close to the same link. We will now define a virtual analog of the Markov moves. The Type I and Type II Markov moves (Definitions 2.2.4 and 2.2.5) remain the same for virtual braids and we will not redefine those moves. We will introduce two new Markov moves, both of which are pictured in Figure 4.6.

Definition 4.1.9. Let $\alpha, \beta \in \mathcal{VB}_n$. A *Type IIIa Markov move*, or a *right virtual exchange move* takes $\alpha\sigma_n\beta\sigma_n^{-1}$ to $\alpha v_n \beta v_n$

Definition 4.1.10. Let $\alpha, \beta \in \mathcal{VB}_n$. Let α' and β' be the images of the embedding of $\mathcal{VB}_n$ to $\mathcal{VB}_{n+1}$ on the last n strands. A *Type IIIb Markov move*, or a *left virtual exchange move* takes $\alpha'\sigma_1\beta'\sigma_1^{-1}$ to $\alpha' v_1 \beta' v_1$

Theorem 4.1.11 (Kamada [17])**.** *Let $\beta \in \mathcal{VB}_n$ and $\beta' \in \mathcal{VB}_{n'}$. β and β' have equivalent closures if and only if we may transform β to β' by a sequence of relations in the virtual braid groups and Markov moves of type I, II, IIIa, and IIIb.*

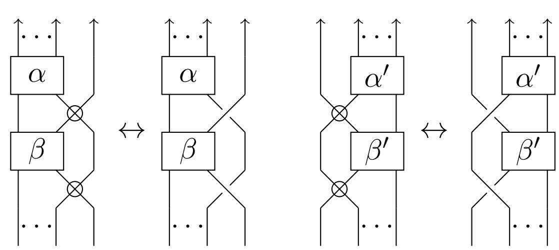


Figure 4.5: Markov Moves of Type IIIa and IIIb

4.2 Standard Bimodules

Standard bimodules are irreducible submodules of Soergel bimodules. We will discuss their importance to studying Soergel bimodule theory and give justification for these bimodules being the "right" categorification of virtual crossings in the category of graded bimodules. For most of this discussion we will follow [41].

Let $R = \mathbb{Q}[x_1, ..., x_n]$, S_n be the nth symmetric group, and let $w \in S_n$. S_n acts on R as before taking x_i to $x_{w(i)}$. Define the *standard bimodule* R_w as an R-module isomorphic to R as a left R-module with right R-action $f \cdot x = w(x)f$. These bimodules are also graded with $\deg_q(x_i) = 2$. We can also describe R_w as a $R \otimes_{\mathbb{Q}} R$-module,

$$R_w \cong \mathbb{Q}[x_1, ..., x_n, y_1, ..., y_n]/(x_1 - y_{w(1)}, ..., x_n - y_{w(n)}).$$

It is easily shown that $\mathbf{p}_w = (x_1 - y_{w(1)}, ..., x_n - y_{w(n)})$ is a regular sequence, so that $K(\mathbf{p}_w)$ is a Koszul resolution for R_w. If $w' \in S_n$ also then we can consider $R_w \otimes_R R'_w$. $R_w \otimes_R R'_w$ is isomorphic to R as a left R-module. If $f \otimes g \in R_w \otimes_R R'_w$ then by definition $(f \otimes g) \cdot x = f \otimes (g \cdot x) = f \otimes w'(x)g = (f \cdot w'(x)) \otimes g = (ww'(x)f) \otimes g = ww'(x)(f \otimes g)$. Therefore $R_w \otimes_R R'_w \cong R_{ww'}$.

Proposition 4.2.1. *Let $w, w' \in S_n$, then $Hom(R_w, R_w) \cong R$ and $Hom(R_w, R_{w'}) = 0$ if $w \neq w'$.*

Proof. Recall that for a commutative ring S, $Hom_S(S/I, S/J) \cong (J : I)/J$ where $(J : I) = \{s \in S | sJ \subset I\}$. If we consider $R_w = (R \otimes_{\mathbb{Q}} R)/(\mathbf{p}_w)$ and $R_{w'} = (R \otimes_{\mathbb{Q}} R)/(\mathbf{p}_{w'})$, then the result follows immediately. □

Let $\mathcal{ST}$ be the full subcategory of graded R-bimodules generated by R_w for all $w \in S_n$ up to direct sums and grading shifts.

Theorem 4.2.2. *$\mathcal{ST}$ is a categorification of the group ring $\mathbb{Z}[q^{\pm 1}][S_n]$.*

Proof. $K_0(\mathcal{ST})$ is a ring with, $[M \otimes_R N] = [M][N]$, since $\mathcal{ST}$ is a monoidal category. We want to define a map $\phi : K_0(\mathcal{ST}) \to \mathbb{Z}[q^{\pm 1}][S_n]$. We will set $\phi(\mathbf{q}^k[R_w]) = q^k w$. Because $R_w \otimes_R R'_w \cong R_{ww'}$, $R_w \otimes_R R_{w^{-1}} \cong R_e$ and $R_{s_i} \otimes_R R_{s_{i+1}} \otimes_R R_{s_i} \cong R_{s_{i+1}} \otimes_R R_{s_i} \otimes_R R_{s_{i+1}}$. Therefore ϕ is a ring morphism. It remains to show that ϕ is invertible, but $\psi(w) = [R_w]$ is an inverse for ϕ as desired. □

Now we will give the justification for the terminology "virtual crossings" in relation to standard bimodules.

Theorem 4.2.3 (Thiel [42]). *Let $R = \mathbb{Q}[x_1, ..., x_n]$ and let $\mathcal{A}$ be the category of graded R-bimodules. Define a map $\mathcal{F} : \mathcal{VB}_n \to \mathcal{K}om(\mathcal{A})$ by the following relations*

$$\mathcal{F}(1) = \quad 0 \longrightarrow R \longrightarrow 0,$$

$$\mathcal{F}(v_i) = \quad 0 \longrightarrow R_{s_i} \longrightarrow 0,$$

$$\mathcal{F}(\sigma_i) = \quad 0 \longrightarrow R \xrightarrow{\chi_+} \mathbf{tq}^{-2} B_i \longrightarrow 0,$$

$$\mathcal{F}(\sigma_i^{-1}) = \quad 0 \longrightarrow \mathbf{t}^{-1} B_i \xrightarrow{\chi_-} R \longrightarrow 0,$$

$$\mathcal{F}(\beta\beta') = \mathcal{F}(\beta) \otimes_R \mathcal{F}(\beta') \textit{ for } \beta, \beta' \in \mathbb{B}_n.$$

Then if β and β' are words representing the same element of $\mathcal{VB}_n$, then $\mathcal{F}(\beta)$ is homotopy equivalent to $\mathcal{F}(\beta')$.

Therefore, in $\mathcal{A}$, the bimodules R_{s_i} represent the action of virtual crossings in the virtual braid group. With this in mind we will now define diagrammatics for the bimodules R_w. First we associate to R_{s_i} the diagram for the elementary virtual crossing v_i. To tensor products $R_w \otimes_R R_{w'}$ we associate concatenation of diagrams as before.

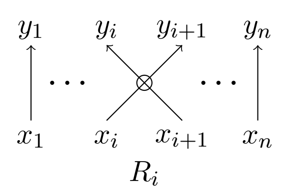


Figure 4.6: Standard bimodule as virtual crossing

4.3 Soergel Bimodules and Filtrations

We now want to introduce how Soergel bimodules and standard bimodules interact. First we consider the Soergel bimodule B_i. For now we restrict ourselves to a certain subclass of Soergel bimodules called Bott-Samelson bimodules.

Definition 4.3.1. Let $w = s_{i_1} \cdots s_{i_k}$ be a (not necessarily reduced) word in S_n. A *Bott-Samelson bimodule* is a Soergel bimodule of the form

$$BS(w) = B_{i_1} \otimes_R \cdots \otimes_R B_{i_k}$$

Remark 4.3.1. *Note that two words, w and w', being equivalent in S_n does not imply that $BS(w) \cong BS(w')$. For example we know from the categorified MOY relations that*

$$BS(s_1 s_1) = B_i \otimes_R B_i \cong B_i \oplus \mathbf{q}^2 B_i \ncong R = BS(1).$$

We will now set $R_i = R_{s_i}$ for brevity. There exists a short exact sequence

$$0 \longrightarrow q^2 R \xrightarrow{x_{i+1}\otimes 1 - 1\otimes x_i} B_i \xrightarrow{1} R_i \longrightarrow 0. \tag{4.1}$$

Therefore, R is a submodule of B_i and $R_i \cong B_i/R$. We can also encode this information in the form of a filtration. By convention, we will consider only finite decreasing filtrations, that is filtrations of the form

$$F(M) : 0 = F_i(M) \subset F_{i-1}(M) \subset \cdots \subset F_{i-k}(M) = M.$$

Also recall the associated graded module of a filtered module is defined as $G^F(M) = \bigoplus_i G_i^F(M)$ where $G_i^F(M) = F_i(M)/F_{i+1}(M)$.

In the case of the short exact sequence (4.1), we may define the *positive filtration* of B_i, $F_1^+(B_i) = \mathbf{q}^2 R$ and $F_0^+(B_i) = B_i$. The associated graded of this filtration

$$G^+(B_i) = G^{F^+}(B_i) = G_1^+(B_i) \oplus G_0^+(B_i).$$

In this case $G_1^+(B_i) = \mathbf{q}^2 R$ and $G_0^+(B_i) = R_i$. We can also consider another short exact sequence

$$0 \longrightarrow q^2 R_i \xrightarrow{x_i\otimes 1 - 1\otimes x_i} B_i \xrightarrow{1} R \longrightarrow 0 \tag{4.2}$$

Therefore, using the exact sequence (4.2), we may define the *negative filtration* of B_i. Let $F_1^-(B_i) =$

$\mathbf{q}^2R_i$ and $F_0^-(B_i) = B_i$. The associated graded of this filtration is

$$G^-(B_i) = G^{F^-}(B_i) = G_1^-(B_i) \oplus G_0^-(B_i).$$

In this case $G_1^-(B_i) = \mathbf{q}^2R_i$ and $G_0^-(B_i) = R$.

We can also define filtrations on any Bott-Samelson bimodule in a similar manner. Recall that if M and N are filtered bimodules, then $M \otimes_R N$ is also a filtered bimodule with

$$F_p(M \otimes_R N) = \sum_{i+j=p} F_i(M) \otimes_R F_j(N).$$

Note that this sum is normally not a direct sum, but rather the sum $\sum_{i+j=p} Im(\iota_{i,j})$ with $\iota_{i,j}$: $F_i(M) \otimes_R F_j(N) \to M \otimes_R N$ being the inclusion map.

Let $w = s_{i_1} \cdots s_{i_k} \in S_n$ and let $\Delta = \{\delta_1, ..., \delta_k\}$ be a sequence of pluses and minuses. Then we can define the filtration F^Δ on $BS(w)$ by

$$F_p^\Delta(BS(w)) = \sum_{j_1+\cdots+j_k=p} (F_{i_1}^{\delta_1}(B_{s_1}) \otimes_R \cdots \otimes_R F_{i_k}^{\delta_k}(B_{s_k}))$$

Example 4.3.2. Let $w = s_2s_1 \in S_3$ and let $\Delta = \{+, -\}$. Then we can define $F^\Delta(BS(w))$ by the following.

$$\begin{aligned}
F_2^\Delta(BS(w)) &= F_1^+(B_1) \otimes_R F_1^-(B_2) = \mathbf{q}^2R \otimes_R \mathbf{q}^2R_2 \cong \mathbf{q}^4R_2 \\
F_1^\Delta(BS(w)) &= F_1^+(B_1) \otimes_R F_0^-(B_2) + F_0^+(B_1) \otimes_R F_1^-(B_2) \\
&= \mathbf{q}^2R \otimes_R B_2 + B_1 \otimes_R \mathbf{q}^2R_2 \cong \mathbf{q}^2B_2 + \mathbf{q}^2B_1 \otimes_R R_2 \\
F_0^\Delta(BS(w)) &= F_0^+(B_1) \otimes_R F_0^-(B_2) = B_1 \otimes_R B_2
\end{aligned}$$

Direct computation and carefully looking at the sum of inclusions for $F_1^\Delta(BS(w))$ tells us that

$$G_2^\Delta(BS(w)) = R_2,\ G_1^\Delta(BS(w)) = R \oplus R_w,\ G_0^\Delta(BS(w)) = R_1.$$

Note that in our example the associated graded bimodule, G^Δ, consists of direct sums and grading shifts of standard bimodules. Soergel in [41] proves that the same holds for Soergel bimodules in general.

Theorem 4.3.3 (Soergel [41]). *Let B_w be the indecomposable Soergel bimodule labelled by $w \in S_n$. Then there exists a unique filtration $F^{std}(B_w)$, called the standard filtration, such that*

$$G_i^{std}(B_w) = \bigoplus_{x \le w, \ell(x)=i} \mathbf{q}^{2i} R_x.$$

Soergel used these filtrations to prove that B_x categorifies the Kazdahn-Lusztig basis of H_n and to give a proof of the Kazdahn-Luzstig positivity conjecture for H_n.

4.4 Some Homological Algebra

In this section we will discuss how we may convert filtrations of bimodules into iterated mapping cones in a derived category. When we have filtrations which have length greater than 2, it becomes more natural to replace iterated mapping cones with convolutions of twisted complexes. We will introduce twisted complexes and their convolutions and give some basic results. Finally we will explore how a special type of homotopy equivalence, deformation retracts, interact with twisted complexes and their convolutions.

Suppose we have a R-bimodule M and a filtration on M of length two. That is

$$0 = F_2(M) \subset F_1(M) \subset F_0(M) = M.$$

Then we can write a short exact sequence

$$0 \longrightarrow F_1(M) \longrightarrow F_0(M) \longrightarrow G_0(M) \longrightarrow 0$$

Let $\mathcal{A}$ denote the category of graded R-bimodules. Let us now consider M as a filtered object in $\mathcal{D}^b(\mathcal{A})$, the bounded derived category of $\mathcal{A}$. Then the short exact sequence above implies that there exists a map $s \in \operatorname{Ext}^1(G_0(M), F_1(M))$ such that $Cone(s) \sim F_0(M)$, where $\sim$ denotes quasi-isomorphism. In this case, because $F_1(M) = G_1(M)$ and $F_0(M) = M$ then we have described M as the mapping cone of its associated graded modules:

$$M \sim Cone(G_0(M) \xrightarrow{s} G_1(M)).$$

When we are using diagrammatics, instead of writing $Cone(s)$ we will draw a box around a short complex as shown in Figure 4.7

$$Cone\left(X \xrightarrow{f} Y\right) = \boxed{X \xrightarrow{f} Y}$$

Figure 4.7: Alternate notation for mapping cones

We want to find a similar way to describe longer filtrations of bimodules in the bounded derived category. Via an inductive argument using short exact sequences,

$$0 \longrightarrow F_p(M) \longrightarrow F_{p-1}(M) \longrightarrow G_p(M) \longrightarrow 0,$$

we see that we can express the filtration as an iterated mapping cone on the associated graded modules

$$\boxed{G_0(M) \longrightarrow \boxed{G_1(M) \longrightarrow \boxed{G_2(M) \longrightarrow \quad \cdots \quad \longrightarrow \boxed{G_{k-1}(M) \longrightarrow G_k(M)}}}}\ .$$

However, iterated mapping cones can be burdensome to work with. We will introduce an alternative way of dealing with these objects known as twisted complexes.

Definition 4.4.1. A *twisted complex* over $\mathcal{D}^b(\mathcal{A})$ is a collection

$$\{X_i, f_{ij} : X_j \to X_i\}_{i \in \mathbb{Z}_{\geq 0}}$$

consisting of finitely many objects $X_i \in \mathcal{D}^b(\mathcal{A})$, $X_0 \neq 0$ and maps $f_{ij} \in \mathrm{Hom}^*(X_j, X_i)$ of homological degree $i - j - 1$ which satisfy the following relations.

- $f_{ij} = 0$ if $j \geq i$.
- $\partial f_{ij} := d_{X_i} f_{ij} + (-1)^{i-j+1} f_{ij} d_{X_j} = \sum_{i>k>j} f_{ik} f_{kj}.$

We may also denote a twisted complex by $(X_\bullet, f_{ij})$, or simply $X_\bullet$ when the maps are understood.

The *length* of a map f_{ij} will be defined as the number $j-i$ and we will define the *length* of a twisted complex by the number maximum number k such that $X_{k-1} \neq 0$. We will now give a couple of basic examples.

Example 4.4.2. Suppose $X_\bullet$ is a twisted complex of length 2. Then we have a twisted complex of the form

$$X_0 \xrightarrow{f_{10}} X_1.$$

The only condition on this twisted complex is that f_{10} is a chain map. Now suppose that $X_\bullet$ is a twisted complex of length 3. Then $X_\bullet$ is a twisted complex of the form

$$\begin{tikzcd} X_0 \arrow[r, "f_{10}"'] \arrow[rr, bend left, "f_{20}"] & X_1 \arrow[r, "f_{21}"'] & X_2 \end{tikzcd}$$

The conditions given in the definition of a twisted complex tell us that $f_{21}f_{10} = \partial f_{20}$. That is $f_{21}f_{10}$ is homotopic to the zero map and f_{20} is the homotopy.

We may also define a category of twisted complexes.

Definition 4.4.3. The category of twisted complexes over $\mathcal{D}^b(\mathcal{A})$, which we will denote by $\mathcal{TW}(\mathcal{A})$, is the category with objects being twisted complexes $(X_\bullet, f_{ij})$ and morphisms described as follows. Let $(X_\bullet, f_{ij})$ and $(Y_\bullet, g_{k\ell})$ be two twisted complexes, then degree k maps are those that intertwine the diagrams formed by A and B,

$$\operatorname{Hom}^k_{\mathcal{TW}(\mathcal{A})}(X_\bullet, Y_\bullet) = \prod_{j \leq i} \operatorname{Hom}^{k+j-i}(X_j, Y_i).$$

Remark. We can turn $\mathcal{TW}(\mathcal{A})$ into a dg-category by the defining a differential d on $\operatorname{Hom}^*_{\mathcal{TW}(\mathcal{A})}(X_\bullet, Y_\bullet)$. If $\phi \in \operatorname{Hom}^*_{\mathcal{TW}(\mathcal{A})}(X_\bullet, Y_\bullet)$ then define

$$(d\phi)_{ij} = (-1)^j \partial(\phi_{ij}) + \sum_k (g_{ik}\phi_{kj} - (-1)^{|\phi|}\phi_{ik} f_{hj}).$$

Direct computation will show that this differential turns $\mathcal{TW}(\mathcal{A})$ into a dg-category [10].

Note that twisted complexes have a natural filtration as follows. If $(X_\bullet, f_{ij})$ is a twisted complex then define the filtration F by $F_m(X_\bullet, f_{ij}) = \{(X_i), f_{ij} : X_j \to X_i\}_{i,j \geq m}$. In particular this means that we replace any complex X_i or map f_{ij} where i or j is smaller than m with the zero complex and map respectively. Then clearly $F_n(X_\bullet) \subset F_{n-1}(X_\bullet) \subset \cdots \subset F_0(X_\bullet) = X_\bullet$ and the maps f_{ij} are filtration preserving maps. Note that any morphism of twisted complexes also respects this filtration by definition.

We will draw twisted complexes like chain complexes, but with extra differentials that "skip" certain complexes.

$$\begin{tikzcd}
X_1 \arrow[r, "f_{21}"] \arrow[rrrr, bend left=40, "f_{n1}"] \arrow[rr, bend right, "f_{31}"'] & X_2 \arrow[r, "f_{32}"] \arrow[rrr, bend left, "f_{n2}"'] & X_3 \arrow[r, "f_{43}"] \arrow[rr, bend right, "f_{n3}"'] & \cdots \arrow[r, "f_{n,n-1}"'] & X_n
\end{tikzcd}$$

Figure 4.8: Example diagram of a twisted complex.

Definition 4.4.4. The convolution of a twisted complex $(X_\bullet, f_{ij})$ is the chain complex in $\mathcal{D}^b(\mathcal{A})$ defined as follows

$$Con(X_\bullet)_i = (X_0)_i \oplus (X_1)_{i+1} \oplus \cdots \oplus (X_n)_{i+n-1} \qquad D = \begin{pmatrix} d_{X_0} & & & & \\ f_{10} & d_{X_1} & & & \\ f_{20} & f_{21} & d_{X_2} & & \\ \vdots & \vdots & \ddots & \ddots & \\ f_{n0} & f_{n1} & \cdots & f_{n,n-1} & d_{X_n} \end{pmatrix}$$

We will denote this chain complex by $Con(X_\bullet, f_{ij})$ or briefly by $Con(X_\bullet)$ when the maps are understood. We may also denote a convolution by drawing a box around the diagram for a twisted complex in a manner similar to mapping cones. (see Figure 4.9)

$$
\begin{tikzcd}
X_1 \arrow[r, "f_{21}"] \arrow[rrrr, bend left=40, "f_{n1}"] \arrow[rr, bend right=40, "f_{31}"'] & X_2 \arrow[r, "f_{32}"] \arrow[rrr, bend left=30, "f_{n2}"] & X_3 \arrow[r, "f_{43}"] \arrow[rr, bend right=40, "f_{n3}"'] & \cdots \arrow[r, "f_{n,n-1}"'] & X_n
\end{tikzcd}
$$

Figure 4.9: Alternate notation for Convolutions

Example 4.4.5. Recall when the length of a twisted complex is 2, it has the form

$$X_0 \xrightarrow{f_{10}} X_1,$$

where f_{10} is a chain map. The convolution of this twisted complex is exactly the mapping cone $Cone(f_{10})$.

Now suppose the length of the twisted complex is 3, then recall $X_\bullet$ has the form.

$$
\begin{tikzcd}
X_0 \arrow[r, "f_{10}"'] \arrow[rr, bend left=20, "f_{20}"] & X_1 \arrow[r, "f_{21}"'] & X_2
\end{tikzcd}
$$

Therefore the convolution $Con(X_\bullet)$ is the chain complex with underlying module $X_0 \oplus X_1[1] \oplus X_2[2]$ and differential

$$D = \begin{pmatrix} d_{X_0} & & \\ f_{10} & d_{X_1} & \\ f_{20} & f_{21} & d_{X_2} \end{pmatrix}.$$

Recall, we can express filtrations of bimodules as iterated mapping cones of the associated graded bimodule. Also recall that the mapping cone operation is associative (see [45]), that is

$$\boxed{X \xrightarrow{f+h} \boxed{Y \xrightarrow{g} Z}} = \boxed{\boxed{X \xrightarrow{f} Y} \xrightarrow{h+g} Z}.$$

However, we can also present this iterated mapping cone as the convolution of a twisted complex of length 3,

$$\boxed{\begin{tikzcd} X \arrow[r, "f"'] \arrow[rr, bend left, "h"] & Y \arrow[r, "g"'] & Z \end{tikzcd}}\ .$$

The above statement is verified by writing down the definition of both the iterated mapping cone and the convolution. Inductively, this implies the following proposition.

Proposition 4.4.1. *Suppose M is an R-bimodule with a filtration of length n. Then there exists a twisted complex $X(M) \in \mathcal{TW}(\mathcal{A})$ of length n such that $Con(X(M)) \sim M$.*

Finally we will address a special type of homotopy equivalence and how it interacts with the concepts of twisted complexes and convolutions. We know from basic topology that a deformation retract is a retract $r : X \to A \subset X$ such that r is homotopic to the identity map on X. The following definition is the equivalent notion in homological algebra.

Definition 4.4.6. Let X and X' be chain complexes. A *deformation retract* is a chain map $\Psi : X \to X'$ such that there exists a reverse map $\Psi' : X' \to X$ and a homotopy $\psi : X \to X$ such that

1. $\Psi'\Psi = Id_X + \partial\psi$.
2. $\Psi\Psi' = Id_{X'}$.
3. $\psi\psi = 0$.
4. $\Psi\psi = 0$.
5. $\psi\Psi' = 0$.

Remark. Actually only the first two conditions are necessary to be a deformation retract. If Ψ, Ψ', and ψ meet conditions (1) and (2) above, then replacing ψ with $\psi' = (\partial\psi)\psi(\partial\psi)d_X(\partial\psi)\psi(\partial\psi)$ will satisfy the last 3 conditions [6]. However, we will always assume that ψ meets all of the conditions if it is the homotopy for a deformation retract.

We now want to see how a deformation retract interacts with a mapping cone. Let $f_{21} : X_1 \to X_2$ be a chain map and suppose that $\Psi_i : X_i \to X'_i$ is a deformation retract with reverse map Ψ'_i and homotopy ψ_i. We want to know if we can use this information to construct a deformation retract $\nabla : Cone(f_{21}) \to Cone(\Psi_2 f_{21} \Psi_1')$.

Consider the square,

$$\begin{tikzcd} X_1 \arrow[r, "f_{21}"] \arrow[d, "\Psi_1"'] \arrow[dr, dashed, "h"] & X_2 \arrow[d, "\Psi_2"] \\ X_1' \arrow[r, "\Psi_2 f_{21} \Psi_1'"'] & X_2'. \end{tikzcd}$$

Since Ψ_1 is a deformation retract, then this square commutes up to homotopy. That is $\Psi_2 f_{21} - \Psi_2 f_{21} \Psi_1' \Psi_1 = \Psi_2 f_{21}(Id_{X_1} - \Psi_1'\Psi_1) = \Psi_2 f_{21} \partial(-\psi_1) = \partial(-\Psi_2 f_{21} \psi_1)$. It is a well-known fact that homotopy communtative squares induce chain maps between mapping cones [45]. Let $h = -\Psi_2 f_{21} \psi_1$. Then this homotopy commutative square gives a map of mapping cones

$$\nabla = \begin{pmatrix} \Psi_1 & 0 \\ h & \Psi_2. \end{pmatrix}.$$

Now we want to construct a candidate for the reverse map $\nabla' : Cone(\Psi_2 f_{21} \Psi_1') \to Cone(f_{21})$. Consider the square,

$$\begin{tikzcd} X_1' \arrow[r, "\Psi_2 f_{21} \Psi_1'"] \arrow[d, "\Psi_1'"'] \arrow[dr, dashed, "h'"] & X_2' \arrow[d, "\Psi_2'"] \\ X_1 \arrow[r, "f_{21}"'] & X_2 \end{tikzcd}$$

Since Ψ_2 is a deformation retract, then this square commutes up to homotopy. That is $f_{21}\Psi_1' - \Psi_2'\Psi_2 f_{21} \Psi_1' = (Id_{X_2} - \Psi_2'\Psi_2) f_{21} \Psi_1' = \partial(\psi_2) f_{21} \Psi_1' = \partial(-\psi_2 f_{21} \Psi_1')$. Let $h' = \psi_2 f_{21} \Psi_1'$. Then this homotopy commutative square gives a map of mapping cones

$$\nabla' = \begin{pmatrix} \Psi_1' & 0 \\ h' & \Psi_2'. \end{pmatrix}.$$

Lemma 4.4.7. *∇ is a deformation retract with reverse map ∇'.*

Proof. First we need to find a homotopy. Let

$$\delta = \begin{pmatrix} \psi_1 & 0 \\ \psi_2 f_{21} \psi_1 & \psi_2 \end{pmatrix}.$$

First we prove condition (2). Note that

$$\nabla\nabla' = \begin{pmatrix} \Psi_1\Psi_1' & 0 \\ h\Psi_1' + \Psi_2 h' & \Psi_2\Psi_2'. \end{pmatrix}.$$

Since Ψ_1 and Ψ_2 are deformation retracts, then $\Psi_i'\Psi_i = Id_{X_i}$, $\Psi_2 h' = \Psi_2\psi_2 f_{21}\Psi_1' = 0$ and $h\Psi_1 = \Psi_2 f_{21}\psi_1\Psi_1 = 0$. Therefore $\nabla\nabla' = Id_{Cone(\Psi_2 f_{21}\Psi_1')}$.

Next we will prove condition (1). Note that

$$\begin{aligned}\nabla'\nabla = \begin{pmatrix} \Psi_1'\Psi_1 & 0 \\ h'\Psi_1' + \Psi_2 h & \Psi_2'\Psi_2. \end{pmatrix} &= \begin{pmatrix} Id_{X_1} + \partial(\psi_1) & 0 \\ \partial(\psi_2 f_{21}\psi_1) & Id_{X_2} + \partial(\psi_2) \end{pmatrix} \\ = \begin{pmatrix} Id_{X_1} & 0 \\ 0 & Id_{X_2} \end{pmatrix} + \partial \begin{pmatrix} \psi_1 & 0 \\ \psi_2 f_{21}\psi_1 & \psi_2 \end{pmatrix}.\end{aligned}$$

Therefore $\nabla'\nabla = Id_{Cone(f_{21})}$ as desired. Conditions (3), (4), and (5) follow from similar computations.

□

Proposition 4.4.8. *Let $X_i \in \mathcal{D}^b(\mathcal{A})$ for $i = 0, 1, ..., n$, and suppose that $\Psi_i : X_i \to X_i'$ is a deformation retract. Suppose that $X_\bullet = (X_i, f_{ij})$ is a twisted complex. Then there exist maps $f_{ij}' : X_j' \to X_i'$ such that $X_\bullet' = (X_i', f_{ij}')$ is a twisted complex and there exists a map $\nabla : Con(X_\bullet) \to Con(X_\bullet')$ such that ∇ is a deformation retract.*

Proof. We prove this by induction on the length of the twisted complex. The case that the twisted complex has length 2 was proven in Lemma 4.4.7. Suppose that the twisted complex $X_\bullet$ has length $k+1$. Then we may consider $Con(X_\bullet)$ as a mapping cone,

$$Cone(X_0 \xrightarrow{\sum_{i=1}^k f_{i1}} Con((X_\ell, f_{ij})_{\ell \geq 1}).$$

By the induction hypothesis, since $Con((X_\ell, f_{ij})_{\ell\geq 1})$ is a convolution of a twisted complex of length k, then there exists a deformation retract $\nabla_1 : Con((X_\ell, f_{ij})_{\ell\geq 1}) \to Con((X'_\ell, f'_{ij})_{\ell\geq 1})$. Therefore by Lemma 4.4.7 there exists a deformation retract,

$$\nabla : Cone(X_0 \xrightarrow{\sum_{i=1}^k f_{i0}} Con((X_\ell, f_{ij})_{\ell\geq 1})) \to Cone(X'_0 \xrightarrow{\sum_{i=1}^k f'_{i0}} Con((X'_\ell, f'_{ij})_{\ell\geq 1})).$$

But, $Cone(X'_0 \xrightarrow{\sum_{i=1}^k f'_{i0}} Con((X'_\ell, f'_{ij})_{\ell\geq 1}) = Con(X'_\bullet)$. □

Corollary 4.4.9. *Let $X_i \in \mathcal{D}^b(\mathcal{A})$ for $i = 0, 1, ..., n$, and suppose that $\Psi_i : X_i \to X'_i$ is a deformation retract. Suppose that $X_\bullet = (X_i, f_{ij})$ is a twisted complex such that $f_{ij} = 0$ when $|i - j| \geq 2$, that is $f_{i+1,i} f_{i,i-1} = 0$. Then there exist maps $f'_{ij} : X'_j \to X'_i$ such that $X'_\bullet = (X'_i, f'_{ij})$ is a twisted complex and there exists a map $\nabla : Con(X_\bullet) \to Con(X'_\bullet)$ such that ∇ is a deformation retract. In particular,*

$$f'_{ij} = \Psi_i f_{i,i-1} \psi_{i-1} f_{i-1,i-2} \psi_{i-2} \cdots f_{j+1,j} \Psi'_j.$$

Proof. We know that there exist maps f'_{ij} by Proposition 4.4.8. We can find the exact formulae by repeatedly applying the computations from Lemma 4.4.7. □

4.5 Soergel Bimodules as Convolutions

Now we want to express B_i and more generally Bott-Samelson bimodules as convolutions of twisted complexes. Recall that we had two filtrations for B_i, F^+ and F^-. By our discussion in Section 4.4, we know that the short exact sequence,

$$0 \longrightarrow F_1^+(B_i) \longrightarrow F_0^+(B_i) \longrightarrow G_0^+(B_i) \longrightarrow 0,$$

and the fact F^+ has length 2, implies that $B_i \sim Cone(G_0^+(B_i) \xrightarrow{S} G_1^+(B_i))$ for some $S \in \mathrm{Ext}^1(G_0^+(B_i), G_1^+(B_i))$. Recall that $G_0^+(B_i) = R_i$ and $G_1^+(B_i) = \mathbf{q}^2 R$. Therefore if we denote B_i^+ as B_i with the positive filtration,

$$B_i^+ \sim_F Cone(R_i \xrightarrow{S} \mathbf{bq}^2 R).$$

We use $\sim_F$ to denote that the quasi-isomorphism is filtered and $\mathbf{b}^k N$ to denote $N \subset G^k(M)$, or that the *filtration degree* of N is k. If we let B_i^- denote B_i with the negative filtration, then we may go through a similar argument to show that

$$B_i^- \sim_F Cone(R \xrightarrow{S'} \mathbf{bq}^2 R_i).$$

Though on general principles we know that such an S and S' exists, it will be useful later to know explicitly how to represent these maps on Koszul resolutions. We will first construct such a map between the Kozsul complexes for R and R_i

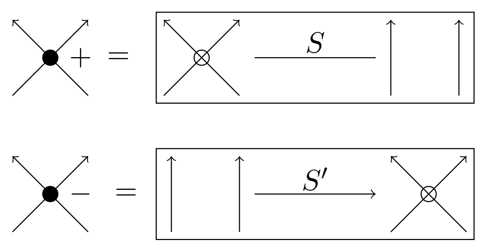


Figure 4.10: Diagrammatic form of B_i^+ and B_i^-

Recall the Koszul resolutions of R and R_i,

$$R \sim \begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 \end{bmatrix}, \; R_i \sim \begin{bmatrix} x_1 - y_2 \\ x_2 - y_1 \end{bmatrix}.$$

Then we can write a map $S \in \mathrm{Ext}^1(R_e, \mathbf{q}^2 R)$ as a chain map with a-degree -1 and b-degree 1 between Koszul complexes. Recall that $R^e = R \otimes_{\mathbb{Q}} R$, then the map S has the presentation

$$
\begin{tikzcd}
\mathbf{q}^4\mathbf{a}^2R^e \arrow[r, "\begin{pmatrix} x_1-y_2 \\ x_2-y_1 \end{pmatrix}"] \arrow[dr, "\begin{pmatrix} 1 \\ -1 \end{pmatrix}"] & \mathbf{q}^2\mathbf{a}R^e \oplus \mathbf{q}^2\mathbf{a}R^e \arrow[r, "\begin{pmatrix} x_2-y_1 & y_2-x_1 \end{pmatrix}"] \arrow[dr, "\begin{pmatrix} 1 & 1 \end{pmatrix}"] & R^e \\
\mathbf{b}\mathbf{q}^6\mathbf{a}^2R^e \arrow[r, "\begin{pmatrix} x_1-y_1 \\ x_2-y_2 \end{pmatrix}"'] & \mathbf{b}\mathbf{q}^4\mathbf{a}R^e \oplus \mathbf{b}\mathbf{q}^4\mathbf{a}R^e \arrow[r, "\begin{pmatrix} x_2-y_2 & y_1-x_1 \end{pmatrix}"'] & \mathbf{b}\mathbf{q}^2R^e.
\end{tikzcd}
$$

We can then look at $Cone(S)$,

$$
\begin{tikzcd}
\mathbf{q}^4\mathbf{a}^2R^e \oplus \mathbf{b}\mathbf{q}^6\mathbf{a}^2R^e \arrow[r, "\begin{pmatrix} x_1-y_2 & 0 \\ x_2-y_1 & 0 \\ 1 & x_1-y_1 \\ -1 & x_2-y_2 \end{pmatrix}"] & \mathbf{q}^2\mathbf{a}R^e \oplus \mathbf{q}^2\mathbf{a}R^e \oplus \mathbf{b}\mathbf{q}^4\mathbf{a}R^e \oplus \mathbf{b}\mathbf{q}^4\mathbf{a}R^e \arrow[d, "\begin{pmatrix} x_2-y_1 & y_2-x_1 & 0 & 0 \\ 1 & 1 & x_2-y_2 & y_1-x_1 \end{pmatrix}"] \\
& R^e \oplus \mathbf{b}\mathbf{q}^2R^e.
\end{tikzcd}
$$

If we use Gaussian elimination on this complex, using the bottom left term of each differential as the isomorphism, then we get the complex,

$$
\mathbf{b}\mathbf{q}^6\mathbf{a}^2R^e \xrightarrow{\begin{pmatrix} x_1+x_2-y_1-y_2 \\ x_1x_2-y_1y_2 \end{pmatrix}} \mathbf{b}\mathbf{q}^4\mathbf{a}R^e \oplus \mathbf{q}^2\mathbf{a}R^e \xrightarrow{\begin{pmatrix} x_1x_2-y_1y_2 & y_1+y_2-x_1-x_2 \end{pmatrix}} R^e.
$$

This is exactly the Koszul resolution of B_i. Therefore this map S is a representative of the map expected above. To find a representative of $S' \in \operatorname{Ext}^1(R, \mathbf{q}^2R_1)$, we just need to transpose y_1 and y_2 in the above argument.

Definition 4.5.1. A *virtual saddle* map, $\Sigma \in \operatorname{Ext}^1(R_w, \mathbf{q}^2R_{w'})$, is a map of the form $Id^{\otimes(i-1)} \otimes S \otimes Id^{\otimes(n-i-1)}$ or $Id^{\otimes(i-1)} \otimes S' \otimes Id^{\otimes(n-i-1)}$.

Before discussing filtrations on Bott-Samelson bimodules in more detail, we also need to discuss the marks which come from gluing diagrams together as mentioned in Section 3.1. These marks serve the purpose of remembering the variables in the intermediate copy of R in the tensor product. If we consider M as a $\mathbb{Q}[x_1, ..., x_n]$-$\mathbb{Q}[y_1, ..., y_n]$-bimodule, and N as a $\mathbb{Q}[y_1, ..., y_n]$-$\mathbb{Q}[z_1, ..., z_n]$-bimodule, then $M \otimes_{\mathbb{Q}[y_1,...,y_n]} N$ is a $\mathbb{Q}[x_1,...,x_n]$-$\mathbb{Q}[z_1,...,z_n]$-bimodule. However we have a choice in how we identify the variables $y_1,...,y_n$ with the set of variables $x_1,...,x_n,z_1,...,z_n$. We may use the relations in M or the relations in N to do the identification. Though this does not affect the isomorphism class of the bimodule, it may affect the explicit differentials in the induced map on Koszul resolutions. Since we have defined our virtual saddle maps in terms of Koszul resolutions, we need to understand how this may affect the differentials.

First let us define the mark removal map Ψ_y. Suppose we have the short chain complex of R-modules

$$R[y] \xrightarrow{y} R[y],$$

then we can split $R[y] \cong R \oplus R\langle y \rangle$. We can then define a map Ψ_y,

$$\begin{tikzcd}
R[y] \arrow[r, "\binom{0}{y}"] \arrow[d, "0"'] & R \oplus R\langle y \rangle \arrow[d, "\begin{pmatrix} 1 & 0 \end{pmatrix}"'] \\
0 \arrow[r, "0"] & R,
\end{tikzcd}$$

and a map Ψ'_y,

$$\begin{tikzcd}
R[y] \arrow[r, "\binom{0}{y}"] & R \oplus R\langle y \rangle \\
0 \arrow[r, "0"] \arrow[u, "0"] & R. \arrow[u, "\binom{1}{0}"']
\end{tikzcd}$$

Finally let ψ_y be the homotopy

$$\begin{tikzcd}
R[y] \arrow[r, "\binom{0}{y}"] & R \oplus R\langle y\rangle \arrow[dl, "\begin{pmatrix} 0 & -1/y \end{pmatrix}"'] \\
R[y] \arrow[r, "\binom{0}{y}"] & R \oplus R\langle y\rangle.
\end{tikzcd}$$

Direct computation of $\Psi'_y\Psi_y$, $\Psi_y\Psi'_y$, $\Psi_y\psi_y$, $\psi_y\Psi'_y$ and $\psi_y\psi_y$ prove the following proposition.

Proposition 4.5.2. *The mark removal map Ψ_y is a deformation retract.*

Example 4.5.3. Consider the following Koszul resolution for the bimodule
$M = \mathbb{Q}[x,y]/(x-y) \otimes_{\mathbb{Q}[y]} \mathbb{Q}[y,z]/(y-z) \cong \mathbb{Q}[x,z]/(x-z)$,

$$K = \begin{bmatrix} x-y \\ y-z \end{bmatrix}.$$

We may identify y with x or with z. If we want to identify y with x, first we apply $\Phi^1_{1\to 2}$ to K to get the Koszul complex

$$\begin{bmatrix} x-y \\ x-z \end{bmatrix}.$$

We can contract the tensor factor $(\mathbf{q}^2\mathbf{a}\mathbb{Q}[x,y] \xrightarrow{x-y} \mathbb{Q}[x,y])$ via the mark removal map Ψ_{x-y}. So we get a deformation retract,

$$\begin{bmatrix} x-y \\ y-z \end{bmatrix} \xrightarrow{\Psi_{x-y}\Phi^1_{1\to 2}} \begin{bmatrix} x-z \end{bmatrix},$$

with reverse map $\Phi^{-1}_{1\to 2}\Psi'_{x-y}$ and homotopy ψ_{x-y}.In a similar manner, we may choose to identify y with z to get a deformation retract,

$$\begin{bmatrix} x-y \\ y-z \end{bmatrix} \xrightarrow{\Psi_{y-z}\Phi^1_{2\to 1}} \begin{bmatrix} x-z \end{bmatrix}.$$

with reverse map $\Phi^{-1}_{2\to 1}\Psi'_{y-z}$ and homotopy ψ_{y-z}

We know by Corollary 4.4.8, that the choice of homotopy for deformation retracts determines the maps f'_{ij} in a deformation retract of convolutions. In Example 4.5.3 we see that we have different choices of deformation retracts from $K(x-y, y-z)$ to $K(x-z)$ with distinct homotopies. A natural question we can ask is how these different choices affect the convolution. In particular, are there situations where making a choice causes extra nonzero differential to arise when another choice would not? Luckily the answer to this question is negative.

We will call a variable y an *internal* variable if it corresponds to a mark which can be removed via mark removal. By definition mark removal is a local process.

Lemma 4.5.4. *Suppose that y is an internal variable for a filtered Bott-Samelson bimodule $BS^{\Delta}(w)$. Then the internal variable y can be removed via mark removal in such a way that there are no nonzero differentials of length greater than 1 in the convolution presentation. The corresponding deformation retract is filtered as well.*

Proof. We can represent any filtered Bott-Samelson bimodule, $BS^{\Delta}(w)$ as a convolution of virtual crossings without extra differentials.

$$Con(G_0^{\Delta}(BS(w)) \to G_1^{\Delta}(BS(w)) \to \cdots \to G_k^{\Delta}(BS(w))).$$

with $G_j^{\Delta}(BS(w))$ isomorphic to a direct sum of tensor products of standard bimodules R_w. By Proposition 4.5.2 we know that removing the internal variable y is a strong deformation retract. We know from Corollary 4.4.9 how to write down an explicit SDR between $Con(G_{\bullet}^{\Delta}(BS(w)), f_{ij})$ and $Con(G_{\bullet}^{\Delta}(BS(w))', g_{ij})$ where $\Psi_j : G_j^{\Delta}(BS(w)) \to G_j^{\Delta}(BS(w))'$ is the mark removal map for y. Therefore for $i - j \geq 2$,

$$g_{ij} = \Psi_i f_{i,i-1} \Psi_{i-1} f_{i-1,i-2} \Psi_{i-2} \cdots \Psi_{j+1} f_{j+1,j} \Psi_j'.$$

$\Psi_j'(G_j^{\Delta}(BS(w))') = G_j^{\Delta}(BS(w))'$. In particular, $y \notin G_j^{\Delta}(BS(w))$. The map $f_{j+1,j}$ is a matrix composed of virtual saddle morphisms. Therefore $y \notin f_{j+1,j}(\Psi_j'(G_j^{\Delta}(BS(w))'))$. We know from Proposition 4.5.2 that $\psi_{j-1} = 1/(y-f)$ where f is a polynomial in variables other than y. Therefore $\psi_{j-1} f_{j+1,j}(\Psi_j'(G_j^{\Delta}(BS(w))')) = 0$. The deformation retract being filtered follows from the fact the maps are induced by maps on the underlying twisted complexes. $\square$

Finally we describe how to write Bott-Samelson bimodules as convolutions of twisted complexes. Recall the Bott-Samelson bimodule associated to a non-reduced word $w = s_{i_1} \cdots s_{i_k} \in S_n$ is defined as $BS(w) = B_{i_1} \otimes_R \cdots \otimes_R B_{i_k}$. Also if we let $\Delta = \{\delta_1, ..., \delta_k\}$ be a sequence of pluses and minuses. Then we can define the filtration F^Δ on $BS(w)$ by

$$F_p^\Delta(BS(w)) = \sum_{j_1+\cdots+j_k=p} (F_{i_1}^{\delta_1}(B_{s_1}) \otimes_R \cdots \otimes_R F_{i_k}^{\delta_k}(B_{s_k})).$$

We will also denote the filtration F^Δ on $BS(w)$ by $BS^\Delta(w)$ or by $BS^{\delta_1\cdots\delta_k}(w)$. In particular, we have chosen a filtration F^{δ_j} for each B_{i_j}. We can rewrite each of these $B_{i_j}^{\delta_j}$ as a mapping cone as discussed earlier in this section. Let us start with this case that $k = 2$ and build the more general case inductively. Let $B_{i_1}^{\delta_1} = Cone(G_0(B_{i_1}^{\delta_1}) \to G_1(B_{i_1}^{\delta_1}))$ and $B_{i_2}^{\delta_2} = Cone(G_0(B_{i_2}^{\delta_2}) \to G_1(B_{i_2}^{\delta_2}))$. A general principle says that if we have a tensor product of mapping cones, we can also consider it as an iterated mapping cone. So we can consider $BS^\Delta(s_{i_1}s_{i_2})$ as an iterated mapping cone

$$Cone\left(G_0(B_{i_1}^{\delta_1}) \otimes_R B_{i_2}^{\delta_2} \to G_1(B_{i_1}^{\delta_1}) \otimes_R B_{i_2}^{\delta_2}\right).$$

However, Proposition 4.4.1 says that we may consider iterated mapping cones as convolutions of twisted complexes. Therefore $BS^\Delta(s_{i_1}s_{i_2})$ can be presented as a convolution of the twisted complex

$$\begin{tikzcd}
G_0(B_{i_1}^{\delta_1}) \otimes_R G_0(B_{i_2}^{\delta_2}) \arrow[r] \arrow[d] & G_0(B_{i_1}^{\delta_1}) \otimes_R G_1(B_{i_2}^{\delta_2}) \arrow[d] \\
G_1(B_{i_1}^{\delta_1}) \otimes_R G_0(B_{i_2}^{\delta_2}) \arrow[r] & G_1(B_{i_1}^{\delta_1}) \otimes_R G_1(B_{i_2}^{\delta_2}).
\end{tikzcd}$$

Note that in this presentation of the convolution, there are no differentials of length greater than 1.

Example 4.5.5. Let $w = s_2s_1 \in S_3$ and let $\Delta = \{+, -\}$. Let us present $BS^{+-}(w)$ as the convolution of a twisted complex. We know that $B_2^+ = Cone(R_2 \xrightarrow{S} \mathbf{bq}^2R)$ and $B_1^- = Cone(R \xrightarrow{S'} \mathbf{bq}^2R_1)$. Therefore $BS^{+-}(w) = B_2^+ \otimes_R B_1^-$ can be presented as a convolution of the twisted complex

$$\begin{tikzcd}
R \otimes_R R_1 \arrow[r, "S' \otimes 1"] \arrow[d, "1 \otimes S"'] & \mathbf{bq}^2R_2 \otimes_R R_1 \arrow[d, "1 \otimes S"] \\
\mathbf{bq}^2R \otimes_R R \arrow[r, "S' \otimes 1"'] & \mathbf{b}^2\mathbf{q}^4R_2 \otimes_R R.
\end{tikzcd}$$

Note that from this twisted complex we can see the associated graded modules for $BS^{+-}(w)$ easily. They are

$$G_0^{+-}(BS(w)) = R \otimes_R R_1$$

$$G_1^{+-}(BS(w)) = \mathbf{bq}^2 R_2 \otimes_R R_1 \oplus \mathbf{bq}^2 R \otimes_R R$$

$$G_2^{+-}(BS(w)) = \mathbf{b}^2\mathbf{q}^4 R_2 \otimes_R R.$$

This verifies the claim made about the associated graded bimodules in Example 4.3.2.

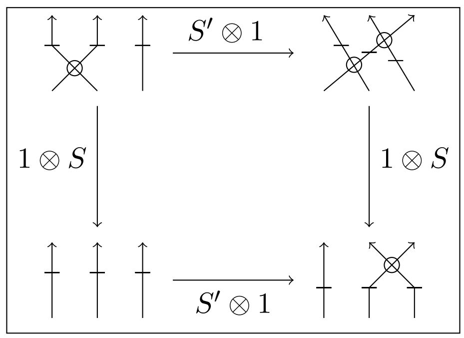


Figure 4.11: Diagrammatic form of Example 4.5.5

Now we can inductively show how to represent any Bott-Samelson bimodule as the convolution of its associated graded bimodules. Let $w = s_{i_1} \cdots s_{i_k}$ and $\Delta = \{\delta_1, ..., \delta_k\}$. We can consider $BS^{\Delta}(w)$ as $BS^{\Delta'}(w') \otimes_R BS^{\delta_{i_k}}(s_{i_k})$, where $w' = s_{i_1} \cdots s_{i_k}$ and $\Delta' = \{\delta_1, ..., \delta_{k-1}\}$. By the induction hypothesis, we can write $BS^{\Delta'}(w')$ as a convolution

$$Con(G_0^{\Delta'}(BS(w')) \to G_1^{\Delta'}(BS(w')) \to \cdots \to G_{k-1}^{\Delta'}(BS(w'))).$$

Therefore we may present $BS^{\Delta}(w)$ as a convolution of mapping cones

$$Con\left(G_0^{\Delta'}(BS(w')) \otimes_R B_{i_k}^{\delta_k} \to G_1^{\Delta'}(BS(w')) \otimes_R B_{i_k}^{\delta_k} \to \cdots \to G_{k-1}^{\Delta'}(BS(w')) \otimes_R B_{i_k}^{\delta_k}\right).$$

Therefore we may consider $BS^{\Delta}(w)$ as a convolution of the twisted complex

$$\begin{tikzcd}
G_0(BS^{\Delta'}(w')) \otimes_R G_0(B_{i_k}^{\delta_k}) \arrow[r] \arrow[d] & G_0(BS^{\Delta'}(w')) \otimes_R G_1(B_{i_k}^{\delta_k}) \arrow[d] \\
G_1(BS^{\Delta'}(w')) \otimes_R G_0(B_{i_k}^{\delta_k}) \arrow[r] \arrow[d] & G_1(BS^{\Delta'}(w')) \otimes_R G_1(B_{i_k}^{\delta_k}). \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
G_k(BS^{\Delta'}(w')) \otimes_R G_0(B_{i_k}^{\delta_k}) \arrow[r] & G_k(BS^{\Delta'}(w')) \otimes_R G_1(B_{i_k}^{\delta_k}).
\end{tikzcd}$$

As before, in this presentation of the convolution, there are no differentials of length larger than 1. To recap, we may build the underlying twisted complexes in the following manner. To each $B_{i_j}^{\delta_j}$ we associate the short twisted complex $G_0^{\delta_j}(B_{i_j}) \xrightarrow{\Sigma} G_1^{\delta_j}(B_{i_j})$ for the appropriate virtual saddle morphism Σ. We then tensor together all of these short twisted complexes together and take their convolution.

Example 4.5.6. Let $w = s_1s_2s_1 \in S_3$ and let $\Delta = \{+,+,+\}$. Let us present $BS^{+++}(w)$ as a convolution of a twisted complex. We know $B_2^+ = Cone(R_2 \xrightarrow{S} \mathbf{bq}^2R)$ and $B_1^+ = Cone(R_1 \xrightarrow{S} \mathbf{bq}^2R)$. Therefore $BS^{+++}(w)$ is the convolution of the twisted complex in Figure 4.12. We have omitted the maps in this complex. Note that from this twisted complex we can see the associated graded modules for $BS^{+++}(w)$ easily. They are

$$G_0^{+++}(BS(w)) = R_2 \otimes_R R_1 \otimes_R R_2$$

$$G_1^{+++}(BS(w)) = (\mathbf{bq}^2R \otimes_R R_1 \otimes_R R_2) \oplus (\mathbf{bq}^2R_2 \otimes_R R \otimes_R R_2) \oplus (\mathbf{bq}^2R_2 \otimes_R R_1 \otimes_R R)$$

$$G_2^{+++}(BS(w)) = (\mathbf{b}^2\mathbf{q}^4R \otimes_R R \otimes_R R_2) \oplus (\mathbf{b}^2\mathbf{q}^4R \otimes_R R_1 \otimes_R R) \oplus (\mathbf{b}^2\mathbf{q}^4R_2 \otimes_R R \otimes_R R_2)$$

$$G_3^{+++}(BS(w)) = \mathbf{b}^3\mathbf{q}^6R \otimes_R R \otimes_R R.$$

In Figure 4.13 we include a diagrammatic form of the convolution of this twisted complex.

$$
\begin{tikzcd}
R_1 \otimes_R R_2 \otimes_R R_1 \arrow[rr] \arrow[dr] \arrow[dd] & & \mathbf{b}\mathbf{q}^2 R_1 \otimes_R R_2 \otimes_R R \arrow[dr] \arrow[dd] & \\
& \mathbf{b}\mathbf{q}^2 R_1 \otimes_R R \otimes_R R_1 \arrow[rr] \arrow[dd] & & \mathbf{b}^2\mathbf{q}^4 R_1 \otimes_R R \otimes_R R \arrow[dd] \\
\mathbf{b}\mathbf{q}^2 R \otimes_R R_2 \otimes_R R_1 \arrow[rr] \arrow[dr] & & \mathbf{b}^2\mathbf{q}^4 R \otimes_R R_2 \otimes_R R \arrow[dr] & \\
& \mathbf{b}^2\mathbf{q}^4 R \otimes_R R \otimes_R R_1 \arrow[rr] & & \mathbf{b}^3\mathbf{q}^6 R \otimes_R R \otimes_R R.
\end{tikzcd}
$$

Figure 4.12: Convolution presentation of $BS^{+++}(w)$.

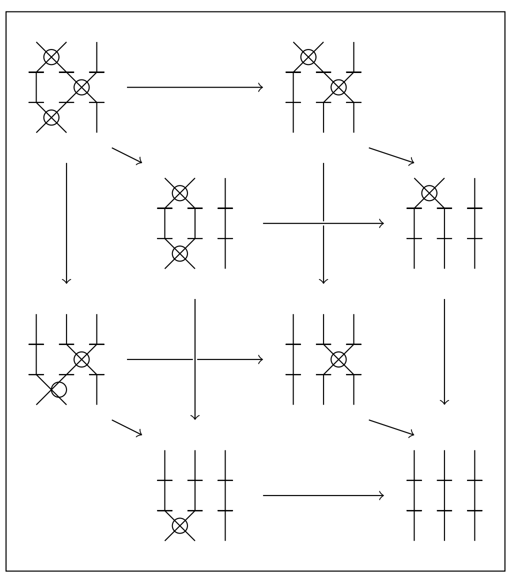


Figure 4.13: Diagrammatic form of Example 4.5.6

CHAPTER 5: Filtered HOMFLY-PT Homology

In this chapter we will introduce definitions for filtered HOMFLY-PT homology using the tools developed in the previous chapter using virtual crossings and filtrations. We will begin by proving some diagrammatic lemmas for virtual crossings that allow us to more easily work with the corresponding objects in $\mathcal{K}om(\mathcal{D}^b(\mathcal{A}))$. Next, after addressing filtered versions of the categorified MOY relations, we will define filtered HOMFLY-PT homology and address Reidemeister II, III and the Markov moves. Though we can prove invariance of Reidemeister II and Markov moves, we can only conjecture invariance for Reidemeister III. However, we will also discuss bounds on how far off Reidemeister III can be from being an invariant.

5.1 Diagrammatic Lemmas for Virtual Crossings

In this section we will explain how virtual saddles interact with mark removal maps. Finally we will address the categorified MOY relations in terms of virtual crossings. We know that mark removal does not affect the differentials of the twisted complexes by Lemma 4.5.4, therefore we will no longer keep track of the exact variables tied to marks. Also we may "slide" marks through virtual crossings as well. For example suppose we have the Koszul complex

$$\begin{bmatrix} x_1 - z_1 \\ x_2 - y \\ y - z_2 \end{bmatrix}.$$

This complex can be represented diagrammatically in two ways as shown below in Figure 5.1. Note that the bimodules these diagrams represent are not just isomorphic, but are equal.

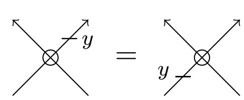


Figure 5.1: Mark sliding diagrammatics

As with mark removal, mark sliding is a local process. Therefore we can always slide marks through any virtual crossings. Next we may ask what happens to the virtual saddle morphisms S and S' when we tensor with other standard bimodules and perform mark removal. First we define a set of virtual saddles Σ_{ij} in the following manner. Let R_w and $R_{w'}$ be standard bimodules for some elements $w, w' \in S_n$ such that $w(i,j) = w'$ where (i,j) is the transposition switching i and j and leaving all other numbers fixed. Suppose that $\mathbf{p}_w = (x_1 - y_{w(1)}, ..., x_n - y_{w(n)})$ and $\mathbf{p}_{w'} = (x_1 - y_{w'(1)}, ..., x_n - y_{w'(n)})$. Then we define $\Sigma_{ij} : K(\mathbf{p}_w) \to \mathbf{bq}^2 K(\mathbf{p}_{w'})$ by the map,

$$\begin{tikzcd}
\begin{bmatrix} x_i - y_{w(i)} \\ x_j - y_{w(j)} \end{bmatrix} = \mathbf{q}^4\mathbf{a}^2 R^e \arrow[r, "\begin{pmatrix} x_i - y_{w(i)} \\ x_j - y_{w(j)} \end{pmatrix}"] \arrow[d] \arrow[dr, "\begin{pmatrix} 1 \\ -1 \end{pmatrix}"] & \mathbf{q}^2\mathbf{a}R^e \oplus \mathbf{q}^2\mathbf{a}R^e \arrow[r, "\begin{pmatrix} x_j - y_{w(j)} & -x_i + y_{w(i)} \end{pmatrix}"] \arrow[dr, "\begin{pmatrix} 1 & 1 \end{pmatrix}"] & R^e \\
\begin{bmatrix} x_i - y_{w'(i)} \\ x_j - y_{w'(j)} \end{bmatrix} = \mathbf{bq}^6\mathbf{a}^2 R^e \arrow[r, "\begin{pmatrix} x_i - y_{w'(i)} \\ x_j - y_{w'(j)} \end{pmatrix}"'] & \mathbf{bq}^4\mathbf{a}R^e \oplus \mathbf{bq}^4\mathbf{a}R^e \arrow[r, "\begin{pmatrix} x_j - y_{w'(j)} & -x_i + y_{w'(i)} \end{pmatrix}"'] & \mathbf{bq}^2 R^e,
\end{tikzcd}$$

on the ith and jth tensor components and the identity on all other tensor components. Note that after reordering tensor components it becomes clear that Σ_{ij} are all virtual saddle maps and also that all virtual saddle maps are sums of maps of the form Σ_{ij}. In particular we can write $S = \Sigma_{12} : R_1 \to \mathbf{q}^2 R$ and $S' = \Sigma_{12} : R \to \mathbf{q}^2 R_1$. We hope that looking at the source and target of the map Σ_{ij} will eliminate any ambiguity in the notation.

Lemma 5.1.1 (Intertwining of saddles)**.** *Consider the map $\Sigma_{i,i+1} : R_i \to \mathbf{q}^2 R$. Then the map $Id \otimes \Sigma_{i,i+1} : R_w \otimes_R R_i \to \mathbf{q}^2 R_w \otimes_R R$, after mark removal, becomes*

$$\Sigma_{w(i)w(i+1)} : R_{ws_i} \to \mathbf{q}^2 R_w.$$

Proof. Consider R, R_i, and R_w as Koszul complexes $K(\mathbf{p}_e), K(\mathbf{p}_{s_1})$, and $K(\mathbf{p}_w)$ respectively where

$$\mathbf{p}_a = (x_1 - y_{a(1)}, ..., x_n - y_{a(n)}).$$

We will use the notation $K(p_a^{x,y})$ to denote that we are using the lists of variables $x_1, ..., x_n$ and $y_1, ..., y_n$ in the Koszul complex. We want to see what happens to the map $\Sigma_{i,i+1} : R_1 \to R$ when apply the functor $R_w \otimes_R \bullet$.

We have a map $Id \otimes \Sigma_{i,i+1} : K(\mathbf{p}_w^{x,y}) \otimes_{\mathbb{Q}[y]} K(\mathbf{p}_{s_i}^{y,z}) \to K(\mathbf{p}_w^{x,y}) \otimes_{\mathbb{Q}[y]} K(\mathbf{p}_e^{y,z})$. For brevity we will use $K(\mathbf{p}_w^{x,y}|\mathbf{p}_{s_i}^{y,z})$ to denote $K(\mathbf{p}_w^{x,y}) \otimes_{\mathbb{Q}[y]} K(\mathbf{p}_{s_i}^{y,z})$, and similarly for $K(\mathbf{p}_w^{x,y}|\mathbf{p}_e^{y,z})$. In both of these Koszul complexes, we have that $y_1, ..., y_m$ are internal variables. We have a composition of maps simplifying $K(\mathbf{p}_w^{x,y}|\mathbf{p}_{s_i}^{y,z})$,

$$K(\mathbf{p}_w^{x,y}|\mathbf{p}_{s_i}^{y,z}) \xrightarrow{\Phi_1} K(\mathbf{p}_{ws_i}^{x,z}|\mathbf{p}_{s_i}^{y,z}) \xrightarrow{\Psi_1} K(\mathbf{p}_{ws_i}^{x,z}).$$

Here $\Phi_1 = \Phi^1_{n+1\to 1} \cdots \Phi^1_{n+n\to n}$ and $\Psi_1 = \Psi_{y_1 - z_{s_i(1)}} \cdots \Psi_{y_n - z_{s_i(n)}}$ is a composition of mark removal maps. We also have a composition of maps simplifying $K(\mathbf{p}_w^{x,y}|\mathbf{p}_e^{y,z})$,

$$K(\mathbf{p}_w^{x,y}|\mathbf{p}_e^{y,z}) \xrightarrow{\Phi_2} K(\mathbf{p}_w^{x,z}|\mathbf{p}_e^{y,z}) \xrightarrow{\Psi_2} K(\mathbf{p}_w^{x,z}).$$

$\Phi_2 = \Phi^1_{n+1\to 1} \cdots \Phi^1_{n+n\to n}$ is a composition of change of basis maps and $\Psi_2 = \Psi_{y_1 - z_1} \cdots \Psi_{y_n - z_n}$ is a composition of mark removal maps. So the twisted complex

$$R_w \otimes_R R_1 \xrightarrow{Id \otimes \Sigma_{i,i+1}} \mathbf{q}^2 R_w \otimes_R R$$

is equivalent to the twisted complex

$$R_{ws_1} \xrightarrow{\Psi_2 \Phi_2 (Id \otimes \Sigma_{i,i+1}) \Phi_1^{-1} \Psi_1'} R_w.$$

Direct computation shows that $\Psi_2 \Phi_2 (Id \otimes \Sigma_{i,i+1}) \Phi_1^{-1} \Psi_1' = \Sigma_{w(i),w(i+1)}$. □

Using our lemmas for mark removal and intertwining of saddles we may more easily describe our twisted complexes. For example, let $w = s_2 s_1 \in S_3$ and let $\Delta = \{+, -\}$. Then we know that $BS^\Delta(w)$ is the convolution of a twisted complex

$$\begin{tikzcd}
R \otimes_R R_1 \arrow[r, "S' \otimes 1"] \arrow[d, "1 \otimes S"'] & \mathbf{bq}^2 R_2 \otimes_R R_1 \arrow[d, "1 \otimes S"] \\
\mathbf{bq}^2 R \otimes_R R \arrow[r, "S' \otimes 1"'] & \mathbf{b}^2\mathbf{q}^4 R_2 \otimes_R R.
\end{tikzcd}$$

By Lemma 4.5.4 we know the mark removal map does not introduce any extra differentials of length greater than 1, and by Lemma 5.1.1 we know exactly what the maps will be after removing the marks,

$$\begin{tikzcd}
R_1 \arrow[r, "\Sigma_{23}"] \arrow[d, "\Sigma_{12}"'] & \mathbf{bq}^2 R_{s_2 s_1} \arrow[d, "\Sigma_{13}"] \\
\mathbf{bq}^2 R \arrow[r, "\Sigma_{23}"'] & \mathbf{b}^2\mathbf{q}^4 R_2.
\end{tikzcd}$$

Note that we could have determined the maps from looking at the underlying permutations of the standard bimodules. For example we know we have a nonzero map $R_1 \to \mathbf{bq}^2 R_{s_2 s_1}$, and we know it is a virtual saddle morphism. Therefore, since $(2,3)s_1 = s_2 s_1$ then we know up to sign that map has to be Σ_{23}.

5.2 Filtered MOY isomorphisms

Now that we have established some diagrammatic lemmas, let us discuss the categorified MOY relations with filtrations. Recall that on Soergel bimodules we have the relation $B_i \otimes_R B_j \cong B_j \otimes_R B_i$ for $|i-j| \geq 2$

Proposition 5.2.1 (Filtered Far Commutativity). *Let $\varepsilon = +$ or $-$ and let $\delta = +$ or $-$. Then $B_i^\varepsilon \otimes_R B_j^\delta \sim_F B_j^\delta \otimes_R B_i^\varepsilon$ for $|i-j| \geq 2$*

Proof. We will prove this for the case that $\varepsilon = \delta = +$. The other cases follow via the same argument with the associated graded objects rearranged. $B_i^+ \otimes B_j^+$ is the convolution of the twisted complex

$$\begin{tikzcd}
R_{s_i s_j} \arrow[r, "\Sigma_{i,i+1}"] \arrow[d, "\Sigma_{j,j+1}"'] & \mathbf{b}\mathbf{q}^2 R_i \arrow[d, "\Sigma_{j,j+1}"] \\
\mathbf{b}\mathbf{q}^2 R_j \arrow[r, "\Sigma_{i,i+1}"'] & \mathbf{b}^2\mathbf{q}^4 R.
\end{tikzcd}$$

Also $B_j^+ \otimes_R B_i^+$ is a convolution of the twisted complex

$$\begin{tikzcd}
R_{s_j s_i} \arrow[r, "\Sigma_{j,j+1}"] \arrow[d, "\Sigma_{i,i+1}"'] & \mathbf{b}\mathbf{q}^2 R_j \arrow[d, "\Sigma_{i,i+1}"] \\
\mathbf{b}\mathbf{q}^2 R_i \arrow[r, "\Sigma_{j,j+1}"'] & \mathbf{b}^2\mathbf{q}^4 R.
\end{tikzcd}$$

By definition, $R_{s_i s_j} = R_{s_j s_i}$. Therefore, the identity morphism is a filtered morphism of these twisted complexes.

$\square$

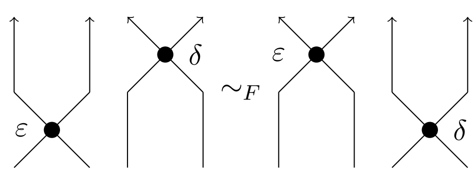


Figure 5.2: Diagrammatic presentation of Proposition 5.2.1

Proposition 5.2.2. *(Filtered MOY II) Let $\varepsilon = +$ or $-$ and let $\delta = +$ or $-$. Then*

$$B_i^{\varepsilon} \otimes_R B_i^{\delta} \sim_F B_i^{c} \oplus \mathbf{b}\mathbf{q}^2 B_i^{c'},$$

where $c = +$ and $c' = -$ if $\varepsilon \neq \delta$ and $c = -$ and $c' = +$ if $\varepsilon = \delta$.

Proof. We will prove this for the case that $\varepsilon = +$ and $\delta = -$. The other cases follow via the same argument with the associated graded objects rearranged. We know that $B_i^+ \otimes_R B_i^-$ is a convolution of the twisted complex

$$R_i \xrightarrow{\begin{pmatrix}\Sigma_{i,i+1}\\ \Sigma_{i,i+1}\end{pmatrix}} \mathbf{bq}^2 R \oplus \mathbf{bq}^2 R_{s_i s_i} \xrightarrow{\begin{pmatrix}\Sigma_{i,i+1} & -\Sigma_{i,i+1}\end{pmatrix}} \mathbf{b}^2\mathbf{q}^4 R_i.$$

Note that $R_{s_i s_i} = R$. Let x and y be the generators for $R \oplus R$. Then we may change the generating set of $\mathbf{bq}^2 R_i \oplus \mathbf{bq}^2 R_i$ from $\{x, y\}$ to $\{x+y, x\}$. This isomorphism becomes an isomorphism on the twisted complex

$$\begin{tikzcd}
R_i \arrow[r, "\begin{pmatrix}\Sigma_{i,i+1}\\ \Sigma_{i,i+1}\end{pmatrix}"] \arrow[d, "1"'] & \mathbf{bq}^2 R \oplus \mathbf{bq}^2 R \arrow[r, "\begin{pmatrix}\Sigma_{i,i+1} & -\Sigma_{i,i+1}\end{pmatrix}"] \arrow[d, "\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)"'] & \mathbf{b}^2\mathbf{q}^4 R_i. \arrow[d, "1"'] \\
R_i \arrow[r, "\begin{pmatrix}\Sigma_{i,i+1}\\ 0\end{pmatrix}"'] & \mathbf{bq}^2 R \oplus \mathbf{bq}^2 R \arrow[r, "\begin{pmatrix}0 & -\Sigma_{i,i+1}\end{pmatrix}"'] & \mathbf{b}^2\mathbf{q}^4 R_i.
\end{tikzcd}$$

The complex splits after the change of generators into two short twisted complexes

$$\left(R_i \xrightarrow{\Sigma_{i,i+1}} \mathbf{bq}^2 R\right) \oplus \left(\mathbf{bq}^2 R \xrightarrow{\Sigma_{i,i+1}} \mathbf{b}^2\mathbf{q}^4 R_i\right).$$

The convolution of the first twisted complex is B_i^+ and the convolution of the second twisted complex is $\mathbf{bq}^2 B_i^-$. Therefore $B_i^+ \otimes_R B_i^- \sim_F B_i^+ \oplus \mathbf{bq}^2 B_i^-$ as desired. $\square$

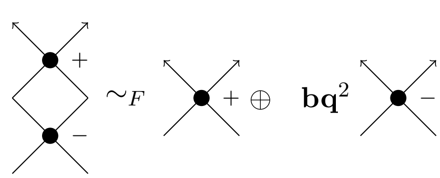


Figure 5.3: Diagrammatic presentation of Proposition 5.2.2

Finally we will discuss a filtered version of the categorified MOY III. This isomorphism unfortunately does not persist through the inclusion of filtrations. However we can at least describe $B_i \otimes_R B_{i+1} \otimes_R B_i$ as a mapping cone in terms of B_i and $B_{i,i+1}$ with the proper filtrations. Recall we let Δ be a sequence of +'s and -'s. We can also consider each δ_i as ± 1 so that we may multiply them. For example if $\delta_1 = +$ and $\delta_2 = -$ then $\delta_1\delta_2 = -$

Proposition 5.2.3. *(Filtered MOY III) Let $\delta_1, \delta_2, \delta_3 \in \{+, -\}$. Then*

$$B_i^{\delta_1} \otimes_R B_{i+1}^{\delta_2} \otimes_R B_i^{\delta_3} \sim_F \begin{cases} Cone(B^*_{i,i+1} \to \mathbf{bq}^2 B_i^{-\delta_1\delta_2\delta_3}) & \text{if } \delta_2 = + \\ Cone(\mathbf{bq}^2 B_i^{-\delta_1\delta_2\delta_3} \to B^*_{i,i+1}) & \text{if } \delta_2 = - \end{cases}$$

Here $B^*_{i,i+1}$ is the filtration induced by the quotient $(B_i^{\delta_1} \otimes_R B_{i+1}^{\delta_2} \otimes_R B_i^{\delta_3})/B_i^{-\delta_1\delta_2\delta_3}$.

Proof. We will present the proof for only the case that $\Delta = \{\delta_1, \delta_2, \delta_3\} = \{+, +, +\}$, as this is the most relevant case to later discussions. The other cases follow by very similar arguments.

We have $B_i^+ \otimes_R B_{i+1}^+ \otimes_R B_i^+$ is the convolution of the twisted complex,

$$\begin{tikzcd}
R_{s_1s_2s_1} \arrow[rr] \arrow[dr] \arrow[dd] & & \mathbf{bq}^2 R_{s_1s_2} \arrow[dr] \arrow[dd] & \\
& \mathbf{bq}^2 R \arrow[rr] \arrow[dd] & & \mathbf{b}^2\mathbf{q}^4 R_1 \arrow[dd] \\
\mathbf{bq}^2 R_{s_2s_1} \arrow[rr] \arrow[dr] & & \mathbf{b}^2\mathbf{q}^4 R_2 \arrow[dr] & \\
& \mathbf{b}^2\mathbf{q}^4 R_1 \arrow[rr] & & \mathbf{b}^3\mathbf{q}^6 R,
\end{tikzcd}$$

where the maps are the appropriate virtual saddle maps with the appropriate signs. The front face, C_f, is the twisted complex for $\mathbf{bq}^2 B_i^+ \otimes_R B_i^-$ and the back face, C_b is the twisted complex for $B_i^+ \otimes_R R_{i+1} \otimes_R B_i^+$. Therefore we may think of this convolution as a mapping cone, $Cone(C_b \xrightarrow{\lambda} C_f)$, by associativity of cones. We know from Proposition 5.2.2 that C_f splits as $\mathbf{bq}^2(B_i^+ \oplus \mathbf{bq}^2 B_i^+)$. Call this splitting isomorphism on $\mathbf{bq}^2 B_i^+ B_i^+$ by Ψ. Then since Ψ is an invertible map, the following square commutes and gives a morphism of mapping cones

$$\begin{tikzcd}
C_b \arrow[r, "\lambda"] \arrow[d, "1"'] & C_f \arrow[d, "\Psi"'] \\
C_b \arrow[r, "\Psi\lambda"] & \mathbf{bq}^2(B_i^+ \oplus \mathbf{bq}^2 B_i^+).
\end{tikzcd}$$

After we apply the morphism of mapping cones, we get a twisted complex of the form

$$\begin{tikzcd}
& R_{s_1 s_2 s_1} \arrow[dl] \arrow[dr] \arrow[drr, dashed, "\Sigma_{13}"] & & \\
\mathbf{bq}^2 R_{s_1 s_2} \arrow[d] \arrow[drr] & & \mathbf{bq}^2 R_{s_2 s_1} \arrow[dll] \arrow[d] \arrow[dr, dashed, "\Sigma_{13}"] & \mathbf{bq}^2 R \arrow[d] \\
\mathbf{b}^2\mathbf{q}^4 R_1 \arrow[dr] & & \mathbf{b}^2\mathbf{q}^4 R_2 \arrow[dl] & \mathbf{b}^2\mathbf{q}^4 R_1 \\
& \mathbf{b}^3\mathbf{q}^6 R. & &
\end{tikzcd}$$

The convolution of this twisted complex can be see as a cone of the dashed maps $\Sigma_{13} : C_1 \to C_2$. C_1 is a convolution presentation of a filtration of $B_{i,i+1}$ and C_2 is a convolution presentation of $\mathbf{bq}^2 B_i^-$. Therefore $B_i^+ B_{i+1}^+ B_i^+ \sim_F Cone(B_{i,i+1}^* \to \mathbf{bq}^2 B_i^-)$. $\square$

Remark. We can explictly describe the possible filtrations on $B_{i,i+1}$ and any other Soergel bimodule parameterized by the longest word w_n in S_n for any integer n. In general there are at least $n^{n-2}n!$ filtrations for the Soergel bimodule B_{w_n}.

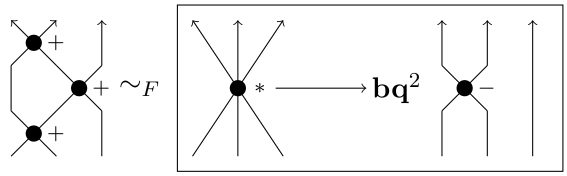


Figure 5.4: Diagrammatic presentation of Proposition 5.2.3

5.3 Filtered HOMFLY-PT Homology

Now that we have developed the appropriate diagrammatic lemmas to work with our virtual crossings and filtrations, we will now define filtered HOMFLY-PT homology. In this section we will give a definition of this homology theory and we will present our results on the behavior of this theory. Conjecturally this will give a link invariant, but we can give bounds on how far off it could possibly be from being an invariant.

Recall that the Rouquier functor $\mathcal{F}$ was defined in terms of how it acted on the elementary braids. In particular,

$$\mathcal{F}(\sigma_i) = \quad 0 \longrightarrow R \xrightarrow{\chi_+} \mathbf{tq}^{-2}B_i \longrightarrow 0,$$

$$\mathcal{F}(\sigma_i^{-1}) = \quad 0 \longrightarrow \mathbf{t}^{-1}B_i \xrightarrow{\chi_-} R \longrightarrow 0.$$

Khovanov and Rozansky showed in [22] that if we consider B_i as a mapping cone, χ_+ and χ_- are homotopy equivalent to the natural inclusion and projections of mapping cones. Define the functor $\mathcal{G}$ by the following relations

$$\mathcal{G}(\sigma_i) = \quad 0 \longrightarrow R \xrightarrow{\chi_+} \mathbf{b}^{-1}\mathbf{tq}^{-2}B_i^+ \longrightarrow 0,$$

$$\mathcal{G}(\sigma_i^{-1}) = \quad 0 \longrightarrow \mathbf{t}^{-1}B_i^- \xrightarrow{\chi_-} R \longrightarrow 0.$$

$\mathcal{G}(1)$ and $\mathcal{G}(\beta\beta')$ are defined as before.

Proposition 5.3.1 ([22])**.** *χ_+ and χ_- are homotopy equivalent to the natural inclusion, and projection respectively, of mapping cones.*

We will from this point forward use χ_+ and χ_- for the natural cone maps as well. Note that the shift in filtration on B_i^+ in $\mathcal{F}'(\sigma_i)$ means that the map χ_+ has filtration grading 0. That is, $\chi_+(x) \in F_0(\mathbf{b}^{-1}B_i^+)$ but $\chi_+(x) \notin F_1(\mathbf{b}^{-1}B_i^+)$.

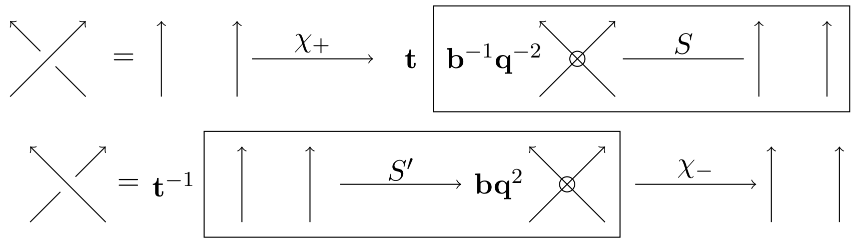


Figure 5.5: Diagrammatic form of $\mathcal{G}(\sigma_i)$ and $\mathcal{G}(\sigma_i^{-1})$.

Before describing the full construction of filtered HOMFLY-PT homology, let us first give a couple of results describing properties of $\mathcal{G}$. We will prove these results in section 5.4

Theorem 5.3.2 (Filtered Reidemeister II)**.** *The Reidemeister II isomorphism is a filtered isomorphism. That is,*

$$\mathcal{G}(\sigma_i\sigma_i^{-1}) \simeq_F \mathcal{G}(1).$$

Reidemeister II holds in the filtered case with no issues. Unfortunately Reidemeister III does not behave as well. Before considering closing the braids the following proposition is as much as we can say.

Proposition 5.3.3. *Let $A = \mathcal{G}(\sigma_i\sigma_{i+1}\sigma_i)$ and A' be the complex shown in Figure 5.6. Then there exists a filtered map $f : A \to A'$ and a collection of maps $g_i : F_i(A') \to F_i(A)$ such that each $f_i = f|_{F_i(A)}$ is a deformation retract with reverse map g_i. The same holds true if A is replaced with $\mathcal{G}(\sigma_{i+1}\sigma_i\sigma_{i+1})$*

Remark. Actually the conditions of Proposition 5.3.3 say something a little stronger about the collection of maps g_i. Suppose $j > i$, then g_j is homotopic to g_i when considered as maps from $F_j(A')$ to $F_i(A)$.

We now give the construction of filtered HOMFLY-PT homology. Let $\mathbf{s} = \mathbf{t}^{-1/2}\mathbf{q}\mathbf{a}^{1/2}$. Define the shifted functor $\widetilde{\mathcal{G}}$ by $\widetilde{\mathcal{G}}(\sigma_i) = \mathbf{s}\mathbf{b}^{1/2}\mathcal{G}(\sigma_i)$ and $\widetilde{\mathcal{G}}(\sigma_i^{-1}) = \mathbf{s}^{-1}\mathbf{b}^{-1/2}\mathcal{G}(\sigma_i^{-1})$. We define $\widetilde{\mathcal{G}}(1)$ and $\widetilde{\mathcal{G}}(\beta\beta')$ as before. We also define a shifted Hochschild homology functor $\widetilde{\mathrm{HH}}(\bullet, R) = \mathbf{s}^n\mathbf{a}^{-n}\mathbf{b}^{n/2}\mathrm{HH}(\bullet, R)$ as before.

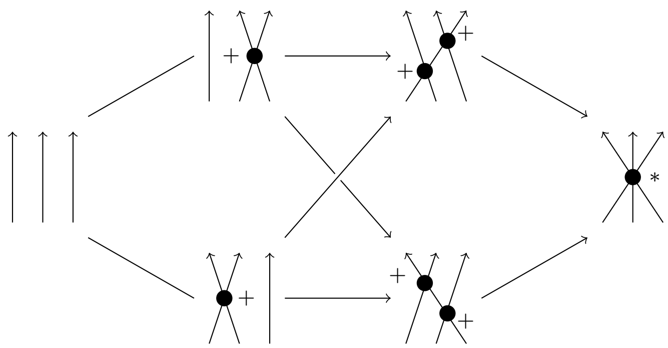


Figure 5.6: Simplified Reidemeister III complex for Proposition 5.3.3.

Now we describe how to compute $\mathcal{H}_F(\beta)$, the filtered version of $\mathcal{H}(\beta)$. We first apply the functor $\widetilde{\mathcal{G}}$ to a braid representative of a link L, β, to receive an object of $\mathcal{K}om(\mathcal{D}^b(\mathcal{A}))$,

$$\cdots \xrightarrow{d} \widetilde{\mathcal{G}}_{j-1}(\beta) \xrightarrow{d} \widetilde{\mathcal{G}}_j(\beta) \xrightarrow{d} \cdots .$$

We then apply our shifted Hochschild homology functor $\widetilde{\mathrm{HH}}(\bullet, R)$ to $\widetilde{\mathcal{G}}(\beta)$ to get a new chain complex of doubly graded vector spaces, $\widetilde{\mathrm{HH}}(\widetilde{\mathcal{G}}(\beta), R)$

$$\cdots \xrightarrow{\widetilde{\mathrm{HH}}(d)} \widetilde{\mathrm{HH}}(\widetilde{\mathcal{G}}_{j-1}(\beta), R) \xrightarrow{\widetilde{\mathrm{HH}}(d)} \widetilde{\mathrm{HH}}(\widetilde{\mathcal{G}}_j(\beta), R) \xrightarrow{\widetilde{\mathrm{HH}}(d)} \cdots .$$

Define $\mathcal{H}_F(\beta) = H(\widetilde{\mathrm{HH}}(\widetilde{\mathcal{G}}(\beta), R))$. Note that if we ignore the filtration on $\mathcal{H}_F(\beta)$, we get back HOMFLY-PT homology, $\mathcal{H}(L)$.

Theorem 5.3.4 (Filtered Markov Moves)**.** *Let $\beta \in \mathcal{B}_n$. The following moves induce filtered isomorphisms on $\mathcal{H}_F(\beta)$.*

I. Braid conjugation on β, that is replacing $\beta = \alpha\alpha'$ with $\alpha'\alpha$.

II. Positive or negative (de)stabilization on β, that is doing the replacement $\beta \mapsto \beta\sigma_n^{\pm 1}$ or vice versa.

We know that Reidemeister II holds and the Markov moves give filtered isomorphisms on $\mathcal{H}_F(\beta)$, finally we want to ask if this means that $\mathcal{H}_F(L)$ is a filtered link invariant. In particular, we want to know that the filtration on $\mathcal{H}_F(L)$ is a link invariant as well as the triply-graded vector space.

Conjecture 5.3.1. *Let L be a link and let β be a braid representative of L. Then $\mathcal{H}_F(\beta)$ is a filtered link invariant whose associated graded gives a quadruply-graded link homology theory.*

This conjecture will immediately follow from Theorem 5.3.2, Theorem 5.3.4, and the following conjecture.

Conjecture 5.3.2. *Let β and β' be two braid representatives of a link L which differ only by a Reidemeister III isotopy (that is the relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$). Then $\mathcal{H}_F(\beta)$ is filtered isomorphic to $\mathcal{H}_F(\beta')$.*

Though we do not yet know for sure that Reidemeister III holds for this link invariant, we have computational evidence that it should hold at least in certain cases. Also there is evidence from other considerations as mentioned in Chapter 1. However, we do know we can bound how far $\mathcal{H}_F(L)$ can be from being a link invariant.

Theorem 5.3.5. *Let β be a braid representative of a link L. Let x be a generator of tri-degree (i, j, k) in $\mathcal{H}_F(\beta)$ with filtration degree m (That is, $x \in F_m(\mathcal{H}_F(\beta))$ but $x \notin F_{m+1}(\mathcal{H}_F(\beta))$). Suppose that β' is another braid representative of L differing from β by a Reidemesiter III isotopy. Then the Reidemeister III map from $\mathcal{H}_F(\beta)$ to $\mathcal{H}_F(\beta')$ changes the filtration degree of x by at most 2 and fixes all other gradings.*

Though this theorem is not as strong as saying that $\mathcal{H}_F(L)$ is a link invariant, it does tell us enough information to make this filtration useful for possibly distinguishing knots. In the section 5.6 we will explore both how to compute this invariant and what information we have thus far.

5.4 Proofs

In this section we present proofs of the results from the previous section. We will also give evidence for our conjectures on Reidemeister III as well. We begin with the proof of Reidemeister II.

Proof of Theorem 5.3.2. This proof follows a similar outline to the proof of Theorem 3.2.2. $\mathcal{G}(\sigma_i\sigma_i^{-1})$ is a complex of the form

$$
\begin{tikzcd}
 & & R \arrow[dr, "\chi_+"] & \\
\mathcal{G}(\sigma_i\sigma_i^{-1}) = & \mathbf{b}^{-1}\mathbf{t}^{-1}B_i^+ \arrow[ur, "\chi_-"] \arrow[dr, "\chi_+"'] & \oplus & \mathbf{t}\mathbf{q}^{-2}B_i^- \\
 & & \mathbf{b}^{-1}\mathbf{q}^{-2}B_i^+ \otimes_R B_i^- \arrow[ur, "\chi_-"'] &
\end{tikzcd}
$$

where χ_+ and χ_- are the natural inclusion and projection of mapping cones. We know from Proposition 5.2.2 that $B_i^+ \otimes_R B_i^- \sim_F B_i^+ \oplus \mathbf{q}^2 B_i^-$. Therefore, $\mathcal{G}(\sigma_i\sigma_i^{-1})$ is filtered homotopy equivalent to the complex

$$
\begin{tikzcd}
 & R \arrow[dr, "\chi_+"] & \\
\mathbf{b}^{-1}\mathbf{t}^{-1}B_i^- \arrow[ur, "\chi_-"] \arrow[dr, "i_2"'] & \oplus & \mathbf{b}^{-1}\mathbf{t}\mathbf{q}^{-2}B_i^+ \\
 & \mathbf{b}^{-1}\mathbf{q}^{-2}B_i^+ \oplus B_i^-. \arrow[ur, "\pi_1"'] &
\end{tikzcd}
$$

Here i_2 is inclusion into the second direct summand and π_1 is projection from the first summand. Therefore by Gaussian elimination, the above complex is filtered homotopy equivalent to

$$0 \to R \to 0 = \mathcal{G}(1)$$

We have proven that $\mathcal{G}(\sigma_i\sigma_i^{-1}) \simeq_F \mathcal{G}(1)$ as desired. $\square$

Next we discuss the proof of the Markov moves for filtered HOMFLY-PT homology. Recall that when $R = \mathbb{Q}[x_1, ..., x_n]$ that $\mathrm{HH}(R, R)$ is isomorphic to the exterior algebra $\Lambda^n(R^n)$. We can also rewrite the algebra $\Lambda^n(R^n)$ as $\mathbb{Q}[x_1, ..., x_n, \theta_1, ..., \theta_n]$, with the relations $\theta_i\theta_j = -\theta_i\theta_j, \theta_i x_j = x_j\theta_i$ and gradings $\deg_q(\theta_i) = 0$ and $\deg_a(\theta_i) = 1$.

We first want to try to understand what happens when we apply the closing functor $\mathcal{C}\ell_j$ to B_i with the positive and negative filtration.

Lemma 5.4.1. *Let $R = \mathbb{Q}[x_1, ..., x_n]$ and let $R' = \mathbb{Q}[x_1, ..., x_{i-1}, x_{i+1}, ..., x_n]$. Then*

$$\mathcal{C}\ell_i(B_i^+) \sim_F R' \oplus \left(\mathbf{bq}^2 R' \otimes_{\mathbb{Q}} \mathbb{Q}[x_i, \theta_i]\right).$$

Proof. First we apply the closing functor $\mathcal{C}\ell_i$ to Koszul complexes for R_i and R to get the Koszul complexes

$$\mathcal{C}\ell_i(R_i) = \begin{bmatrix} x_i - y_{i+1} \\ x_{i+1} - x_i \end{bmatrix}, \quad \mathcal{C}\ell_i(R) = \begin{bmatrix} 0 \\ x_{i+1} - y_{i+1} \end{bmatrix}.$$

Here we omit rows of the Koszul complex which are not relevant to the virtual saddle map $\Sigma_{i,i+1}$. Note that now we are considering these as complexes of R'-bimodules. So then x_i is an internal variable and can be removed via mark removal. We can simplify $\mathcal{C}\ell_i(R_i)$ by doing a change of basis and doing mark removal on $x_i - y_{i+1}$,

$$\begin{bmatrix} x_i - y_{i+1} \\ x_{i+1} - x_i \end{bmatrix} \xrightarrow{\Phi^1_{1\to 2}} \begin{bmatrix} x_i - y_{i+1} \\ x_{i+1} - y_{i+1} \end{bmatrix} \xrightarrow{\Psi_{x_i - y_{i+1}}} \begin{bmatrix} x_{i+1} - y_{i+1} \end{bmatrix}.$$

The Koszul complex $[x_{i+1} - y_{i+1}]$ is quasi-isomorphic to R' once we include the omitted rows, and the Koszul complex $[0]$ is quasi-isomorphic to $\mathbb{Q}[x_i, \theta_i]$. Also the composition $\Psi_{x_i - y_{i+1}} \Phi^1_{1\to 2}$ is a deformation retract. Therefore by Lemma 4.4.7 we have a deformation retract of mapping cones,

$$\begin{tikzcd}
R_i \arrow[r, "\Sigma_{i,i+1}"] \arrow[d, "\Psi_{x_i - y_{i+1}} \Phi^1_{1\to 2}"'] \arrow[dr, dashed, "h"] & \mathbf{bq}^2 R \arrow[d, "1"] \\
R' \arrow[r, "\Sigma_{i,i+1} \Phi^{-1}_{1\to 2} \Psi'_{x_i - y_{i+1}}"'] & \mathbf{bq}^2 R' \otimes_Q \mathbb{Q}[x_i, \theta_i].
\end{tikzcd}$$

Finally we need to compute the composition $\Sigma_{i,i+1} \Phi^{-1}_{1\to 2} \Psi'_{x_i - y_{i+1}}$. Let $R'^e = R' \otimes_{\mathbb{Q}} R'$, then we have the composition

$$
\begin{tikzcd}
\begin{bmatrix} x_{i+1}-y_{i+1} \end{bmatrix} = \arrow[d, "\Psi'_{x_i-y_{i+1}}"'] & & R' \arrow[r] \arrow[d, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"'] & R'^e \arrow[d, "1"'] \\
\begin{bmatrix} x_i-y_{i+1} \\ x_{i+1}-y_{i+1} \end{bmatrix} = \arrow[d, "\Phi^{-1}_{1\to 2}"'] & R^e \arrow[r] \arrow[d, "1"'] & R^e \oplus R^e \arrow[r] \arrow[d, "\left(\begin{smallmatrix}1 & 0\\-1 & 1\end{smallmatrix}\right)"'] & R^e \arrow[d, "1"'] \\
\begin{bmatrix} x_i-y_{i+1} \\ x_{i+1}-x_i \end{bmatrix} = \arrow[d, "\Sigma_{i,i+1}"'] & R^e \arrow[r] \arrow[dr, "\left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right)"] & R^e \oplus R^e \arrow[r] \arrow[dr, "\left(\begin{smallmatrix}1 & 1\end{smallmatrix}\right)"] & R^e \\
\begin{bmatrix} 0 \\ x_{i+1}-y_{i+1} \end{bmatrix} = & R^e \arrow[r] & R^e \oplus R^e \arrow[r] & R^e.
\end{tikzcd}
$$

Therefore, the composition $\Sigma_{i,i+1}\Phi^{-1}_{1\to 2}\Psi'_{x_i-y_{i+1}}$ is zero which implies that the cone

$$R' \xrightarrow{\Sigma_{i,i+1}\Phi^{-1}_{1\to 2}\Psi'_{x_i-y_{i+1}}} \mathbf{bq}^2 R' \otimes_Q \mathbb{Q}[x_i, \theta_i]$$

splits as $R' \oplus \mathbf{bq}^2 R' \otimes_Q \mathbb{Q}[x_i, \theta_i]$. $\square$

Lemma 5.4.2. *Let $R = \mathbb{Q}[x_1, ..., x_n]$ and let $R' = \mathbb{Q}[x_1, ..., x_{i-1}, x_{i+1}, ..., x_n]$. Then*

$$\mathcal{C}\ell_i(B_i^-) \sim_F Cone\left((R' \otimes_{\mathbb{Q}} \mathbb{Q}[x_i, \theta_i]) \xrightarrow{\tilde{m}} \mathbf{bq}^2 R' \right).$$

Proof. This proof is very similar to the proof of Lemma 5.4.1. We already know that we have a deformation retract,

$$\begin{bmatrix} x_i - y_{i+1} \\ x_{i+1} - x_i \end{bmatrix} \xrightarrow{\Phi^1_{1\to 2}} \begin{bmatrix} x_i - y_{i+1} \\ x_{i+1} - y_{i+1} \end{bmatrix} \xrightarrow{\Psi_{x_i - y_{i+1}}} \begin{bmatrix} x_{i+1} - y_{i+1} \end{bmatrix},$$

which induces a deformation retract of mapping cones,

$$\begin{tikzcd}
R \arrow[rr, "\Sigma_{i,i+1}"] \arrow[d, "1"'] \arrow[drr, dashed, "h"] & & \mathbf{bq}^2 R_i \arrow[d, "\Psi_{x_i - y_{i+1}} \Phi^1_{1\to 2}"] \\
R' \otimes_Q \mathbb{Q}[x_i, \theta_i] \arrow[rr, "\Psi_{x_i - y_{i+1}} \Phi^1_{1\to 2} \Sigma_{i,i+1}"'] & & \mathbf{bq}^2 R'.
\end{tikzcd}$$

The composition $\Psi_{x_i - y_{i+1}} \Phi^1_{1\to 2} \Sigma_{i,i+1}$ is nonzero, and we set it equal to $\tilde{m}$. □

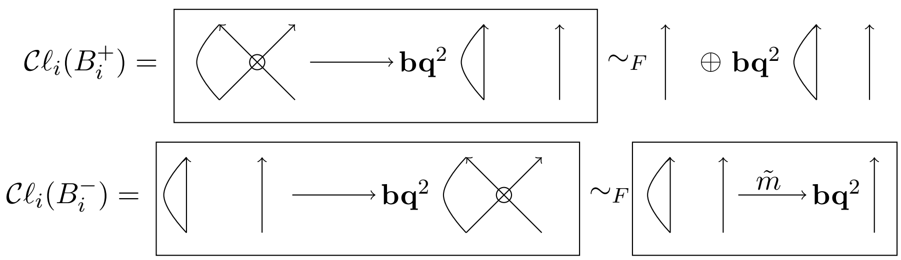


Figure 5.7: Closing functor applied to B_i^+ and B_i^-.

Remark. As can be seen in Figure 5.7, the isomorphism $\mathcal{C}\ell_i(R_i)$ diagrammatically looks like the virtual Reidemeister I move. We have used isomorphisms $R_i \otimes_R R_i \sim R$ and $R_i \otimes_R R_{i+1} \otimes_R R_i \sim R_{i+1} \otimes_R R_i \otimes_R R_{i+1}$ which represent virtual Reidemeister II and III respectively. Also it is not hard to see that semivirtual Reidemeister move holds as well. (see [42] for example). It is natural to ask if the Markov moves of type III hold as well so that $\mathcal{H}(L)$ is a virtual link invariant. Unfortunately as defined this seems not to be the case.

Proof of Theorem 5.3.4. First we address Markov moves of type I. Let α and β be n-braids. Then we want to prove that $\widetilde{\mathrm{HH}}(\widetilde{\mathcal{G}}(\alpha\beta)) \simeq_F \widetilde{\mathrm{HH}}(\widetilde{\mathcal{G}}(\beta\alpha))$. The number of positive and negative crossings in $\alpha\beta$ and $\beta\alpha$ are the same, so it will suffice to show that $\mathrm{HH}(\mathcal{G}(\alpha\beta)) \simeq_F \mathrm{HH}(\mathcal{G}(\beta\alpha))$. By Theorem 3.4.3, we know that $\mathrm{HH}(A \otimes_R B, R) \cong \mathrm{HH}(B \otimes_R A, R)$. Therefore, $\mathrm{HH}(\mathcal{G}_i(\alpha\beta)) \cong \mathrm{HH}(\mathcal{G}_i(\beta\alpha))$ for all i. This isomorphism is clearly filtered and commutes with the differentials by construction.

Now we will prove invariance of $\mathcal{H}_F(\beta)$ under Markov moves of type II. Let $\beta \in \mathcal{B}_n$, and let β'

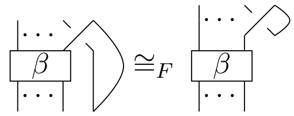


Figure 5.8: Diagrammatics for $\mathcal{C}\ell_{n+1}(\mathcal{G}(\beta'\sigma_n^{\pm 1})) \simeq_F \mathcal{G}(\beta) \otimes_{\mathbb{Q}[x_n]} \mathcal{C}\ell_{n+1}(\mathcal{G}(\sigma_n^{\pm 1}))$.

be its image in $\mathcal{B}_{n+1}$. Then a direct calculation shows that

$$\mathcal{C}\ell_{n+1}(\mathcal{G}(\beta'\sigma_n^{\pm 1})) \simeq_F \mathcal{G}(\beta) \otimes_{\mathbb{Q}[x_n]} \mathcal{C}\ell_{n+1}(\mathcal{G}(\sigma_n^{\pm 1})).$$

This above calculation is visualized in Figure 5.8. Therefore it will suffice to prove the statements for $\beta = 1$. First we address positive stabilization. We know by Lemma 5.4.1 that $\mathcal{C}\ell_{n+1}(B_n^+) \sim_F R' \oplus \left(\mathbf{b}\mathbf{q}^2 R' \otimes_{\mathbb{Q}} \mathbb{Q}[x_{n+1}, \theta_{n+1}]\right)$. Therefore

$$\begin{aligned} \mathcal{C}\ell_{n+1}(\mathcal{G}(\sigma_n)) = \mathcal{C}\ell_{n+1}(R) &\xrightarrow{\chi_i} \mathcal{C}\ell_{n+1}(\mathbf{t}\mathbf{b}^{-1}\mathbf{q}^{-2}B_i^+). \\ \simeq_F R' \otimes_{\mathbb{Q}} \mathbb{Q}[x_{n+1}, \theta_{n+1}] &\xrightarrow{\iota_2} \mathbf{t}\mathbf{b}^{-1}\mathbf{q}^{-2}R' \oplus \mathbf{t}\left(R' \otimes_{\mathbb{Q}} \mathbb{Q}[x_{n+1}, \theta_{n+1}]\right) \end{aligned}$$

By Gaussian elimination, this complex is homotopy equivalent to $\mathbf{t}\mathbf{b}^{-1}\mathbf{q}^{-2}R'$. Since the Gaussian elimination map in this case is a projection and $\mathbf{t}\mathbf{b}^{-1}\mathbf{q}^{-2}R'$ is the highest level of the filtration, then the reverse map and homotopy are filtered by construction. Therefore $\mathcal{C}\ell_{n+1}(\sigma_n) \simeq_F \mathbf{t}\mathbf{b}^{-1}\mathbf{q}^{-2}R' = \mathbf{t}\mathbf{b}^{-1}\mathbf{q}^{-2}\mathcal{G}(1)$, where 1 is the identity braid on n strands. If we replace $\mathcal{G}$ with $\widetilde{\mathcal{G}}$ then $\mathcal{C}\ell_{n+1}(\widetilde{\mathcal{G}}(\sigma_n)) \simeq_F \widetilde{\mathcal{G}}(1)$ as desired.

Finally we look at negative stabilization.

$$\begin{aligned} \mathcal{C}\ell_{n+1}(\mathcal{F}'(\sigma_n^{-1})) = \mathcal{C}\ell_{n+1}(\mathbf{t}^{-1}B_i^-) &\xrightarrow{\chi_o} \mathcal{C}\ell_{n+1}(R) \\ \simeq_F \mathbf{t}^{-1}Cone\left(R' \otimes_{\mathbb{Q}} \mathbb{Q}[x_{n+1}, \theta_{n+1}] \xrightarrow{\tilde{m}} \mathbf{b}\mathbf{q}^2 R'\right) &\xrightarrow{\chi_o} R' \otimes_{\mathbb{Q}} \mathbb{Q}[x_{n+1}, \theta_{n+1}] \end{aligned}$$

The map $\tilde{m}$ is a surjection of a-degree -1. Therefore

$$\mathbf{t}^{-1}Cone\left(R' \otimes_{\mathbb{Q}} \mathbb{Q}[x_{n+1}, \theta_{n+1}] \xrightarrow{\tilde{m}} \mathbf{b}\mathbf{q}^2 R'\right)$$

is filtered quasi-isomorphic to $\mathbf{t}^{-1}R' \otimes_{\mathbb{Q}} \mathbb{Q}\langle x_{n+1}\rangle$. Since χ_o was the natural projection of a mapping cone, then we have a short chain complex.

$$\mathbf{t}^{-1}R' \otimes_{\mathbb{Q}} \mathbb{Q}[x_{n+1}] \xrightarrow{\iota} R' \otimes_{\mathbb{Q}} \mathbb{Q}[x_{n+1}, \theta_{n+1}], \tag{5.1}$$

where ι is the inclusion map. This chain complex is homotopy equivalent to $R' \otimes_{\mathbb{Q}} \mathbb{Q}\theta_{n+1} \cong \mathbf{a}R'$. Note that short chain complex (5.1) only sits in one level of the filtration. Therefore the homotopy equivalence must be filtered by construction. Therefore we have a filtered map Ψ from $\mathcal{G}(\sigma_n^{-1})$ to $\mathbf{a}R'$ and a filtered reverse map Ψ' from $\mathbf{a}R'$ to $\mathcal{G}(\sigma_n^{-1})$ which are homotopy inverses of each other. If we replace $\mathcal{G}$ with $\widetilde{\mathcal{G}}$ then we can replace $\mathbf{a}R'$ with R'.

Finally suppose we have a braid $\beta'\sigma_n^{-1}$ in $\mathcal{B}_{n+1}$ with $\beta' \in \mathcal{B}_n$. Then we have a filtered map $\Psi : \mathcal{C}\ell_{n+1}(\widetilde{\mathcal{G}}(\beta'\sigma_n^{-1})) \to \mathcal{C}\ell_{n+1}(\widetilde{\mathcal{G}}(\beta))$ with filtered reverse map Ψ' such that Ψ is a homotopy equivalence. After applying the rest of the closing functors to compute Hochschild homology and apply the homology functor to the resulting complex, the images of the maps Ψ and Ψ' are still filtered maps. Also $\mathcal{H}_F(\beta'\sigma_n^{-1})$ and $\mathcal{H}_F(\beta)$ are filtered vector spaces. Therefore we have a filtered isomorphism from $\mathcal{H}_F(\beta'\sigma_n^{-1})$ to $\mathcal{H}_F(\beta)$.

$\square$

Finally we address Reidemeister III. First we want to discuss what we can prove and then address what is left to be done. Recall that Proposition 5.2.3 tells how we may rewrite $B_i^+ \otimes_R B_{i+1}^+ \otimes_R B_i^+$ as a mapping cone of $B_{i,i+1}^*$ and B_i^-. We first need to see how close we are to this mapping cone splitting in a filtered manner.

Recall that we have two filtrations on B_i which are not equivalent in a filtered manner. However we know that $B_i^+ = Cone(R_i \xrightarrow{S} \mathbf{bq}^2R)$ and $B_i^- = Cone(R \xrightarrow{S'} \mathbf{bq}^2R)$ are quasi-isomorphic in a non-filtered manner. We may explictly write down the quasi-isomorphism as a map of convolutions after shifting the filtration of one of either B_i^+ or B_i^-. This map, which we will denote as Ξ, is

$$\begin{tikzcd}
\mathbf{b}^{-1}B_i^+ = Con(\mathbf{b}^{-1}R_i \arrow[r, "S"] \arrow[dr, dashed, "h"] & \mathbf{q}^2R \arrow[r] \arrow[d, "\mu^+"] \arrow[dr, dashed, "h'"] & 0) \\
B_i^- = Con(\quad 0 \arrow[r] & R \arrow[r, "S'"] & \mathbf{bq}^2R_i).
\end{tikzcd}$$

μ^+ is multiplication by $x_i - y_{i+1}$ and the dashed arrows are homotopies such that $\partial h = (x_i - y_{i+1})S$ and $\partial h' = S'(x_i - y_{i+1})$. Note that $\chi_o \Xi \chi_i = x_i - y_{i+1}$. Note that is still holds if we take a lower non-trivial level of the filtration of $\mathbf{b}^{-1}B_i^+$. There is also a similar map from $\mathbf{b}^{-1}B_i^-$ to B_i^+, which we will also call Ξ in abuse of notation, where we replace μ^+ with μ^-. Here μ^- is multiplication by $x_i - y_i$.

Lemma 5.4.3. *$F_p(B_i^+ \otimes_R B_{i+1}^+ \otimes_R B_i^+)$ is homotopy equivalent to $F_p(B_{i,i+1}^*) \oplus F_p(\mathbf{bq}^2 B_i^-)$ for all p.*

Proof. We will prove this by ultimately using properties of the map Ξ. We omit grading shifts in this discussion. Consider the diagram of maps.

$$\begin{tikzcd}
B_i^- \arrow[r, "\iota_2"] & B_i^+ \oplus B_i^- \arrow[r, "\chi_i"] \arrow[d, dashed, "x_{i+1} - z"] & B_i^+ \otimes_R B_{i+1}^+ \otimes_R B_i^+ \arrow[d, "Id \otimes \Xi \otimes Id"] \\
B_i^+ \arrow[u, "\Xi"] & B_i^+ \oplus B_i^- \arrow[l, "\pi_1"'] & B_i^+ \otimes_R B_{i+1}^- \otimes_R B_i^+ \arrow[l, "\chi_o"']
\end{tikzcd}$$

We ultimately want to show that $\Xi\pi_1\chi_o(Id \otimes \Xi \otimes Id)\chi_i\iota_2$ is homotopy equivalent to the identity map. Suppose so, then we know that B_i^- is a direct summand of $B_i^+ \otimes_R B_{i+1}^+ \otimes_R B_i^+$. By applying filtered MOY III to $B_i^+ \otimes_R B_{i+1}^+ \otimes_R B_i^+$ to transform it into $Cone(B_{i,i+1}^* \to B_i^-)$, we see that $B_i^+ \otimes_R B_{i+1}^+ \otimes_R B_i^+$ splits into B_i^- and $B_{i,i+1}^*$. Since $\chi_i \Xi \chi_o$ is multiplication on every level of the filtration which is nonzero, we will see that we have a splitting on each level of the filtration.

Now we prove $\Xi\pi_1\chi_o(Id \otimes \Xi \otimes Id)\chi_i\iota_2$ is homotopic to the identity. First we want to discuss $(Id \otimes \Xi \otimes Id)$. Recall $B_i^- \oplus B_i^+$ is filtered isomorphic to $B_i^+ \otimes_R B_i^+$. The map χ_i is really the map $\chi_i : B_i^+ \otimes_R R \otimes_R B_i^+ \to B_i^+ \otimes_R B_i^+ \otimes_R B_i^+$. Therefore the composition $\chi_o(Id \otimes \Xi \otimes Id)\chi_i$ is multiplication by $x_{i+1} - z$, where z is an internal variable. When we split $B_i^+ \otimes_R B_i^+$ back into $B_i^- \oplus B_i^+$, the map $x_{i+1} - z$ becomes the map

$$B_i^+ \oplus B_i^- \xrightarrow{\begin{pmatrix} -z & x_{i+1} \\ 0 & z \end{pmatrix}} B_i^+ \oplus B_i^-.$$

Therefore $\Xi\pi_1\chi_o(Id \otimes \Xi \otimes Id)\chi_i\iota_2 = \Xi z$. This is homotopy equivalent to the identity map up after a q-grading shift.

$\square$

Now we know that every level of $F_p(B_i^+ \otimes_R B_{i+1}^+ \otimes_R B_i^+)$ splits, but the splitting map violates homotopy. In particuar, on the cone presentation of Filtered MOY III, we have the map,

$$Cone(B^*_{i,i+1} \to \mathbf{b}\mathbf{q}^2 B_i^-) \xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ h & 1 \end{smallmatrix}\right)} B^*_{i,i+1} \oplus \mathbf{b}\mathbf{q}^2 B_i^-,$$

where h is a filtration violating homotopy.

Proof of Proposition 5.3.3. We will use the shorthand notation MN for $M \otimes_R N$. Recall $\mathcal{G}(\sigma_i\sigma_{i+1}\sigma_i)$ is the complex

$$\begin{tikzcd}
R \arrow[rr] \arrow[dr] \arrow[dd] & & \mathbf{b}^{-1}\mathbf{t}\mathbf{q}^{-2}B_i^+ \arrow[dr] \arrow[dd] & \\
& \mathbf{b}^{-1}\mathbf{t}\mathbf{q}^{-2}B_i^+ \arrow[rr] \arrow[dd] & & \mathbf{b}^{-2}\mathbf{t}^2\mathbf{q}^{-4}B_i^+B_i^+ \arrow[dd] \\
\mathbf{b}^{-1}\mathbf{t}\mathbf{q}^{-2}B_{i+1}^+ \arrow[rr] \arrow[dr] & & \mathbf{b}^{-2}\mathbf{t}^2\mathbf{q}^{-4}B_{i+1}^+B_i^+ \arrow[dr] & \\
& \mathbf{b}^{-2}\mathbf{t}^2\mathbf{q}^{-4}B_i^+B_{i+1}^+ \arrow[rr] & & \mathbf{b}^{-3}\mathbf{t}^3\mathbf{q}^{-6}B_i^+B_{i+1}^+B_i^+.
\end{tikzcd}$$

By Proposition 5.2.2, We know that $B_i^+B_i^+$ is filtered quasi-isomorphic to $B_i^- \oplus \mathbf{b}\mathbf{q}^2 B_i^+$. Therefore applying this equivalence, filtered MOY III, and applying Gaussian elimination, we get the complex

$$
\begin{tikzcd}
 & & \mathbf{b}^{-1}\mathbf{t}^2\mathbf{q}^{-}4B_i^- \arrow[ddr] & \\
 & \mathbf{t}\mathbf{q}^{-2}B_i \arrow[r] \arrow[ddr] & \mathbf{t}^2\mathbf{q}^{-4}B_iB_{i+1} \arrow[dr] & \\
R \arrow[ur] \arrow[dr] & & & \mathbf{t}^3\mathbf{q}^{-6}Cone(B^*_{i,i+1} \to \mathbf{b}\mathbf{q}^2B_i^-) \\
 & \mathbf{t}\mathbf{q}^{-2}B_{i+1} \arrow[r] \arrow[uur] & \mathbf{t}^2\mathbf{q}^{-4}B_{i+1}B_i \arrow[ur] &
\end{tikzcd}
$$

$Cone(B^*_{i,i+1} \to \mathbf{b}\mathbf{q}^{-2}B_i^-)$ splits in a filtration violating manner via the map

$$Cone(B^*_{i,i+1} \to \mathbf{b}\mathbf{q}^2B_i^-) \xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ h & 1 \end{smallmatrix}\right)} B^*_{i,i+1} \oplus \mathbf{b}\mathbf{q}^2B_i^-.$$

Let f be the composition of the following maps.

$$
\begin{tikzcd}
R \arrow[r] \arrow[d, "1"'] & B_i \oplus B_{i+1} \arrow[r] \arrow[d, "1"'] & B_iB_{i+1} \oplus B_{i+1}B_i \oplus B_i \arrow[r] \arrow[d, "1"'] & Cone(B^*_{i,i+1} \to \mathbf{b}\mathbf{q}^2B_i^-) \arrow[d, "\left(\begin{smallmatrix} 1 & 0 \\ h & 1 \end{smallmatrix}\right)"'] \\
R \arrow[r] \arrow[d, "1"'] & B_i \oplus B_{i+1} \arrow[r] \arrow[d, "1"'] & B_iB_{i+1} \oplus B_{i+1}B_i \oplus B_i \arrow[r] \arrow[d, "\pi"'] & B^*_{i,i+1} \oplus \mathbf{b}\mathbf{q}^2B_i^- \arrow[d, "\pi"'] \\
R \arrow[r] & B_i \oplus B_{i+1} \arrow[r] & B_iB_{i+1} \oplus B_{i+1}B_i \arrow[r] & B^*_{i,i+1}.
\end{tikzcd}
$$

Note that f is a filtered map because the only filtration violating component of the map, h, is not present in the composition. Let g be the compositon of the following maps

$$\begin{tikzcd}
R \arrow[r] \arrow[d, "1"'] & B_i \oplus B_{i+1} \arrow[r] \arrow[d, "1"'] & B_iB_{i+1} \oplus B_{i+1}B_i \arrow[r] \arrow[d, "\iota"'] & B^*_{i,i+1}. \arrow[d, "\left(\begin{smallmatrix} 1 \\ h \end{smallmatrix}\right)"'] \\
R \arrow[r] \arrow[d, "1"'] & B_i \oplus B_{i+1} \arrow[r] \arrow[d, "1"'] & B_iB_{i+1} \oplus B_{i+1}B_i \oplus B_i \arrow[r] \arrow[d, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ -h\chi & -h\chi \end{smallmatrix}\right)"'] & B^*_{i,i+1} \oplus \mathbf{bq}^2B_i^- \arrow[d, "\left(\begin{smallmatrix} 1 & 0 \\ -h & 1 \end{smallmatrix}\right)"'] \\
R \arrow[r] & B_i \oplus B_{i+1} \arrow[r] & B_iB_{i+1} \oplus B_{i+1}B_i \oplus B_i \arrow[r] & Cone(B^*_{i,i+1} \to \mathbf{bq}^2B_i^-)
\end{tikzcd}$$

However, h appears in this composition, therefore g is not a filtered map.

We know from Lemma 5.4.3 that $Cone(F_p(B^*_{i,i+1}) \to F_p(\mathbf{bq}^2B_i^-))$ splits as $F_p(B_{i,i+1}) \oplus F_p(\mathbf{bq}^2B_i^-)$. However the homotopy involved in the splitting, which we will denote by h_p, is not filtered in general. We may repeat the above argument replacing $\mathcal{F}(\sigma_i\sigma_{i+1}\sigma_i)$ with $F_p(\mathcal{F}(\sigma_i\sigma_{i+1}\sigma_i))$ and h with h_p to reach the desired result.

□

We know that now there exists a deformation retract at every level of the filtration on $\mathcal{F}(\sigma_i\sigma_{i+1}\sigma_i)$. However this is not necessarily enough to guarantee that f induces a filtered isomorphism on $\mathcal{H}_F(\beta)$.

Proof of Proposition 5.3.5. We will consider the map $h : B^*_{i,i+1} \to \mathbf{bq}^2B_i^-$ in the splitting of $B_i^+B_{i+1}^+B_i^+ \sim_F Cone(B^*_{i,i+1} \to \mathbf{bq}^2B_i^-)$. We know from our discussion of the proof of Proposition 5.3.3 that h is not filtration preserving. So let us look carefully at the map h. Recall that $B^*_{i,i+1}$ and $\mathbf{bq}^2B_i^-$ have the following twisted complex representations. The condition that h violates filtration means that it may have components of negative b-degree. But because of the structure of $B^*_{i,i+1}$ and $\mathbf{bq}^2B_i^-$, as seen in Figure 5.9, a component of h can have b-degree of -2 at lowest. In other words, h can violate the filtration by at most 2 levels. Let $f : \mathcal{G}(\sigma_i\sigma_{i+1}\sigma_i) \to A'$, where A' is the complex in Figure 5.6, with reverse map g. Also let $f' : \mathcal{G}(\sigma_{i+1}\sigma_i\sigma_{i+1}) \to A'$ with reverse map g'. Then we have a homotopy equivalence $g'f : \mathcal{G}(\sigma_i\sigma_{i+1}\sigma_i) \to \mathcal{G}(\sigma_{i+1}\sigma_i\sigma_{i+1})$. f is filtered, and g' has components which contain h which we know can violate filtration by at most 2 levels. Therefore the map $g'f$ can violate filtration by at most 2 levels. □

$$
\begin{tikzcd}
 & R_{s_1s_2s_1} \arrow[dl] \arrow[dr] & & \\
\mathbf{bq}^2R_{s_1s_2} \arrow[d] \arrow[drr] & & \mathbf{bq}^2R_{s_2s_1} \arrow[dll] \arrow[d] & \mathbf{bq}^2R \arrow[d] \\
\mathbf{b}^2\mathbf{q}^4R_1 \arrow[dr] & & \mathbf{b}^2\mathbf{q}^4R_2 \arrow[dl] & \mathbf{b}^2\mathbf{q}^4R_1 \\
 & \mathbf{b}^3\mathbf{q}^6R. & &
\end{tikzcd}
$$

Figure 5.9: Convolution presentations of $B^*_{i,i+1}$ and $\mathbf{bq}^2B_i^-$

5.5 Computation of $\mathcal{H}_F(\beta)$ for $(2,n)$-Torus Links

Finally we present computations for $\mathcal{H}_F(\beta)$ when $\beta = \sigma_1^n \in \mathcal{B}_2$. We will first write out the case when $n = 2$ without using the shifted functors, as we can just apply the shifts to the end result. We will write out how the complex simplifies in this case so that the process will be clear in later cases. We know from definition $\mathcal{G}(\beta)$ is the complex

$$
\begin{tikzcd}
R \arrow[r, "\chi_i"] \arrow[d, "\chi_i"'] & \mathbf{tq}^{-2}\mathbf{b}^{-1}B_i^+ \arrow[d, "\chi_i"'] \\
\mathbf{tq}^{-2}\mathbf{b}^{-1}B_i^+ \arrow[r, "\chi_i"] & \mathbf{t}^2\mathbf{q}^{-4}\mathbf{b}^{-2}B_i^+ \otimes_R B_i^+.
\end{tikzcd}
$$

$\mathbf{q}^4\mathbf{b}^{-2}B_i^+ \otimes_R B_i^+$ splits as $\mathbf{q}^4\mathbf{b}^{-2}B_i^- \oplus \mathbf{q}^{-2}\mathbf{b}^{-1}B_i^+$. So then $\mathcal{G}(\beta)$ is filtered homotopy equivalent to the complex

$$
\begin{tikzcd}
R \arrow[r, "\chi_i"] \arrow[d, "\chi_i"'] & \mathbf{t}\mathbf{q}^{-2}\mathbf{b}^{-1}B_i^+ \arrow[d, "\begin{pmatrix} * & 1 \end{pmatrix}"'] \\
\mathbf{t}\mathbf{q}^{-2}\mathbf{b}^{-1}B_i^+ \arrow[r, "\begin{pmatrix} * \\ 1 \end{pmatrix}"] & \mathbf{t}^2\mathbf{q}^{-4}\mathbf{b}^{-2}B_i^- \oplus \mathbf{t}^2\mathbf{q}^{-2}\mathbf{b}^{-1}B_i^+.
\end{tikzcd}
$$

This complex can be reduced via Gaussian elimination to the complex

$$R \xrightarrow{\chi_i} \mathbf{t}\mathbf{q}^{-2}\mathbf{b}^{-1} \xrightarrow{\rho^+} \mathbf{t}^2\mathbf{q}^{-4}\mathbf{b}^{-2}B_i^-,$$

where ρ^+ is the map shown below in Figure 5.10.

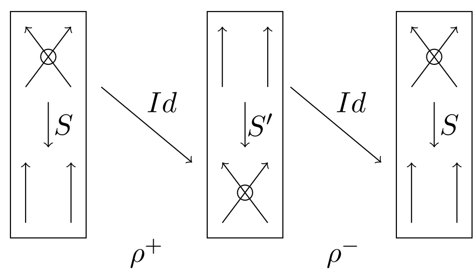


Figure 5.10: Descriptions of the maps ρ^+ and ρ^-.

Let $\mathcal{O} = \mathbb{Q}[x,\theta]$. Now we apply the Hochschild homology functor $\mathcal{G}(\beta)$. From Lemma 5.4.1 and 5.4.2 we know that $\mathrm{HH}(B_i^+) = \mathcal{O} \oplus \mathbf{b}\mathbf{q}^2\mathcal{O}^{\otimes 2}$ and $\mathrm{HH}(B_i^-) = Cone(\mathcal{O}^{\otimes 2} \xrightarrow{m} \mathbf{b}\mathbf{q}^2\mathcal{O})$, where m is multiplication on polynomials in x composed with $\frac{\partial}{\partial\theta} \otimes 1$. Therefore $\mathcal{G}(\beta)$ splits into two subcomplexes; A contractible subcomplex,

$$R \xrightarrow{1} \mathbf{t}R,$$

and a complex

$$\mathbf{t}\mathbf{b}^{-1}\mathbf{q}^{-2}R_i \xrightarrow{\iota} \mathbf{t}^2\mathbf{b}^{-2}\mathbf{q}^{-4}\mathrm{HH}(B_i^-).$$

However, the map m is surjective, so then the complex above simplifies to

$$\mathbf{t}\mathbf{b}^{-1}\mathbf{q}^{-2}\mathcal{O} \xrightarrow{0} \mathbf{t}^2\mathbf{b}^{-2}\mathbf{q}^{-4}\ker(m).$$

The Poincare series of $\mathcal{O}$ is $\dfrac{1+aq^2}{1-q^2}$. So we only need to compute the Poincare series of $\ker(m)$ to finish computing $\mathcal{H}_F(\beta)$. Direct computation shows that the Poincare series of $\ker(m)$ is

$$\frac{(1+aq^2)(1+aq^4)}{(1-q^2)^2}.$$

Therefore, the Poincare series of the unshifted filtered HOMFLY-PT homology, , which we will denote by $\widehat{\mathcal{H}_F}(\beta)$, of the closure of the $(2,2)$-torus braid is

$$\begin{aligned}\mathcal{P}(\sigma_1^2) &= tb^{-1}q^{-2}\frac{1+aq^2}{1-q^2} + t^2b^{-2}q^{-4}\frac{(1+aq^2)(1+aq^4)}{(1-q^2)^2} \\ &= \frac{1+aq^2}{1-q^2}\left[tb^{-1}q^{-2} + t^2b^{-2}q^{-4}\frac{(1+aq^4)}{(1-q^2)}\right].\end{aligned}$$

Now we look at the case $\beta = \sigma_1^3$. Following the same process as before, we get that $\mathcal{G}(\beta)$ is the complex

$$R \xrightarrow{\chi_i} \mathbf{t}\mathbf{b}^{-1}\mathbf{q}^{-2}B_i^+ \xrightarrow{\rho^+} \mathbf{t}^2\mathbf{b}^{-2}\mathbf{q}^{-4}B_i^- \xrightarrow{\rho^-} \mathbf{t}^3\mathbf{b}^{-3}\mathbf{q}^{-6}B_i^+,$$

where ρ^- is shown in Figure 5.10. If we apply the Hochschild homology functor to this complex it splits into three subcomplexes; a contractible subcomplex as before, a complex

$$\mathbf{t}\mathbf{b}^{-1}\mathbf{q}^{-2}\mathcal{O} \xrightarrow{0} \mathbf{t}^2\mathbf{b}^{-2}\mathbf{q}^{-4}ker(m) \xrightarrow{\iota} \mathbf{t}^3\mathbf{b}^{-2}\mathbf{q}^{-4}\mathcal{O}^{\otimes 2},$$

and a complex $\mathbf{t}^3\mathbf{b}^{-3}\mathbf{q}^{-6}\mathcal{O}$. $\mathbf{t}^2\mathbf{b}^{-2}\mathbf{q}^{-4}ker(m) \xrightarrow{\iota} \mathbf{t}^3\mathbf{b}^{-2}\mathbf{q}^{-4}\mathcal{O}^{\otimes 2}$ is an injective map, therefore the homology of this short complex is

$$\mathbf{a}\mathbf{t}^3\mathbf{b}^{-2}\mathbf{q}^{-4}\mathcal{O}^{\otimes 2}/ker(m) \cong \mathbf{t}^3\mathbf{b}^{-2}\mathbf{q}^{-4}\mathcal{O}.$$

Therefore the unshifted filtered HOMFLY-PT homology is isomorphic to $\mathbf{t}\mathbf{b}^{-1}\mathbf{q}^{-2}\mathcal{O} \oplus \mathbf{a}\mathbf{t}^3\mathbf{b}^{-2}\mathbf{q}^{-4}\mathcal{O} \oplus \mathbf{t}^3\mathbf{b}^{-3}\mathbf{q}^{-6}\mathcal{O}$ and therefore the Poincare series of β is

$$\mathcal{P}(\sigma_1^3) = \frac{1+aq^2}{1-q^2}\left[tb^{-1}q^{-2} + at^3b^{-2}q^{-4} + t^3b^{-3}q^{-6}\right].$$

We now give the computation of the unshifted filtered HOMFLY-PT homology for the $(2,n)$-torus braid. Let $\mathbf{s} = \mathbf{t}\mathbf{b}^{-1}\mathbf{q}^{-2}$, and suppose n is even, then the complex $\mathcal{G}(\sigma_i^n)$ is

$$R \xrightarrow{\chi_i} \mathbf{s}B_i^+ \xrightarrow{\rho^+} \mathbf{s}^2 B_i^- \to \cdots \to \mathbf{s}^n B_i^-.$$

After applying the Hochschild homology functor, this complex splits into three types of subcomplexes. First a contractible complex $R \xrightarrow{1} \mathbf{t}R$. Next we have subcomplexes of the form

$$\mathbf{s}^{2k+1}\mathcal{O} \oplus \mathbf{s}^{2k+2}\ker(m) \xrightarrow{\iota} \mathbf{s}^{2k+2}\mathbf{t}^1\mathcal{O}^{\otimes 2}$$

for $k = 0, ..., n/2-2$, which we know from the computation for the $(2,3)$-torus braid is isomorphic to $\mathbf{s}^{2k+1}\mathcal{O} \oplus \mathbf{a}\mathbf{t}\mathbf{s}^{2k+2}\mathcal{O}$. Finally we have a subcomplex of the form

$$\mathbf{s}^{n-1}\mathcal{O} \to \mathbf{s}^n \mathrm{HH}(B_i^-),$$

which we know from the computation for the $(2,2)$-torus braid is isomorphic $\mathbf{s}^{n-1}\mathcal{O} \oplus \mathbf{s}^n ker(m)$. Combining everything, we can have that the Poincare series unfiltered HOMFLY-PT homology of the (2,n)-torus braid with n even is

$$\frac{1+aq^2}{1-q^2}\left[\sum_{i=0}^{n/2-2}(t^{2k+1}b^{-2k-1}q^{-4k-2} + at^{2k+3}b^{-2k-2}q^{-4k-4}) + t^{n-1}b^{1-n}q^{2-2n} + t^n b^{-n} q^{-2n}\frac{1+aq^4}{1-q^2}\right].$$

Finally we address the unshifted HOMFLY-PT homology of the $(2,n)$-torus braid when n is odd. $\mathcal{G}(\sigma_i^n)$ is now the complex

$$R \xrightarrow{\chi_i} \mathbf{s}B_i^+ \xrightarrow{\rho^+} \mathbf{s}^2 B_i^- \to \cdots \to \mathbf{s}^n B_i^+.$$

After applying the Hochschild homology functor, this complex splits into three types of subcomplexes. First a contractible complex $R \xrightarrow{1} \mathbf{t}R$. Next we have subcomplexes of the form

$$\mathbf{s}^{2k+1}\mathcal{O} \oplus \mathbf{s}^{2k+2}\ker(m) \xrightarrow{\iota} \mathbf{s}^{2k+2}\mathbf{t}^1\mathcal{O}^{\otimes 2}$$

for $k = 0, ..., n/2-1$, which we know from the computation for the (2,3)-torus braid is isomorphic to $\mathbf{s}^{2k+1}\mathcal{O} \oplus \mathbf{ats}^{2k+2}\mathcal{O}$. Finally we have a direct summand of the form $\mathbf{s}^n\mathcal{O}$. By combining everything we have that the Poincare series of the unshifted filtered HOMFLY-PT homology of the (2,n)-torus braid when n is odd is

$$\frac{1+aq^2}{1-q^2}\left[\sum_{i=0}^{n/2-1}(t^{2k+1}b^{-2k-1}q^{-4k-2}+at^{2k+3}b^{-2k-2}q^{-4k-4})+t^nb^{-n}q^{-2n}\right].$$

We summarize the results of this section in the following proposition. Let $\widehat{\mathcal{H}_F}(\beta)$ denote the unshifted filtered HOMFLY-PT homology of the braid β and $\mathcal{P}(\beta)$ be its Poincare series.

Proposition 5.5.1. *Let $\beta = \sigma_1^n \in \mathcal{B}_2$ be the (2,n)-torus braid, and let $\mathcal{P}'(\beta) = \dfrac{1+aq^2}{1-q^2}\mathcal{P}(\beta)$. Also let $s = tb^{-1}q^{-2}$, then*

$$\mathcal{P}'(\beta) = \begin{cases} \displaystyle\sum_{i=0}^{n/2-2}(s^{2k+1}+ats^{2k+2})+s^{n-1}+s^n\frac{1+aq^4}{1-q^2} & \textit{if } n \textit{ is even,} \\ \displaystyle\sum_{i=0}^{n/2-1}(s^{2k+1}+ats^{2k+2})+s^n & \textit{if } n \textit{ is odd.} \end{cases}$$

We end this section with a remark about the structure of $\mathcal{P}'(\beta)$. Suppose β is the $(2,2n+1)$-torus braid, then each monomial of $\mathcal{P}'(\beta)$ is of the form $a^it^jb^kq^\ell$ where $\ell = k-j+i$. That is if we factor out the homology of the unknot, $\mathcal{O}$, then the gradings of the generators of $\widehat{\mathcal{H}_F}(\beta)$ are linearly dependent. However in the case that β is the $(2,2n)$-torus braid, then this is not the case. In particular we can see this in the highest t-degree term of $\mathcal{P}'(\beta)$.

CHAPTER 6: Future Work

In this chapter we will discuss future projects which will be based off of this work. Gorsky, Gukov, and Stosic in [16] conjectured a construction of quadruply-graded link homology categorifying the colored HOMFLY-PT polynomial. We conjecture that our construction will correspond to their computations when generalized in the appropriate manner. Also we will discuss how our construction of filtered HOMFLY-PT homology could give rise to an invariant of transverse links.

6.1 Colored Filtered HOMFLY-PT Homology

It has been conjectured by Mackaay, Stosic, and Vaz that an algebric construction of colored HOMFLY-PT homology, in the case that the strands are colored with fundamental representations $\bigwedge^n V$ of $\mathfrak{sl}_m$ for arbitrarily high m, could be given by associating singular Soergel bimodules to the relevant MOY graphs [29]. Williamson and Webster have given a geometric construction of this [43, 44], but as to date no algebraic construction has been given. As shown by Williamson in [46], singular Soergel bimodules also have a standard filtration. Recall that in the uncolored case, the filtrations of Soergel bimodules is are ion in terms of Soergel bimodules and standard bimodules.

If one can give a description of the $(\bigwedge^{n_1} V, ..., \bigwedge^{n_m} V)$-colored HOMFLY-PT homology in terms of filtrations of singular Soergel bimodules in a similar fashion to what we described in Chapters 4 and 5, then the proof of Conjecture 5.3.1. could lead to a filtered theory for HOMFLY-PT homology when we color by the fundamental representations.

If we could find a filtered theory for HOMFLY-PT homology colored with fundamental representations, then we should next try to categorify the highest weight projectors. If we could do this, then we would be able to categorify HOMFLY-PT homology when we color by any finite dimensional representation of $\mathcal{U}_q(\mathfrak{sl}_m)$ for arbitrarily high m. Let V_{ω_i} be the irreducible representation corresponding to the fundamental weight ω_i of $\mathcal{U}_q(\mathfrak{sl}_m)$ for arbitrarily high m.

Cautis in [8] shows that the following process categorifies the highest weight projector for $V_{\omega_{i_1}} \otimes \cdots \otimes V_{\omega_{i_k}}$. Label k strands with $V_{\omega_{i_1}}, \cdots, V_{\omega_{i_k}}$ from left to right. Then take the image of the

(k, pk)-torus braid, $(\sigma_1 \cdots \sigma_{k-1})^{pk} \in \mathcal{B}_k$, in the homotopy category of singular Soergel bimodules and shift the complex so that the identity braid has degree 0 in all degrees, call this complex $C(k, pk)$. It is easy to show that $C(k, pk) \subset C(k, p'k)$ for $p' > p$. Define $P_{i_1,\ldots,i_k} = \lim\limits_{\rightarrow} C(k, pk)$.

Theorem 6.1.1 (Cautis, [8])**.** *The complex $P_{i_1,\ldots,i_k}$ categorifies the highest weight projector*

$$p_{i_1,\ldots,i_k} : V_{\omega_{i_1}} \otimes \cdots \otimes V_{\omega_{i_k}} \to V_{\omega_{i_1}+\cdots+\omega_{i_k}}.$$

The natural question to ask is to see if the filtrations can be seen in these projectors as well. We will give an explicit construction of $P_{1,1}$ in terms of filtrations. By Theorem 6.1.1, we know that we need to construct the "infinite torus braid" on two strands in the uncolored case. In section 5.5, we computed $C(2, 2k)$ to give a method for computing the filtered HOMFLY-PT homology of the (2,2k)-torus link. Recall that if we let $\mathbf{s} = \mathbf{t}\mathbf{b}^{-1}\mathbf{q}^{-2}$, then

$$C(2, 2k) = R \xrightarrow{\chi_i} \mathbf{s}B_i^+ \xrightarrow{\rho^+} \mathbf{s}^2 B_i^- \to \cdots \to \mathbf{s}^{2k} B_i^-.$$

Therefore, $P_{1,1}$ is the 2-periodic infinite complex

$$R \xrightarrow{\chi_i} \mathbf{s}B_i^+ \xrightarrow{\rho^+} \mathbf{s}^2 B_i^- \xrightarrow{\rho^-} \mathbf{s}^3 B_i^+ \xrightarrow{\rho^+} \mathbf{s}^4 B_i^- \to \cdots.$$

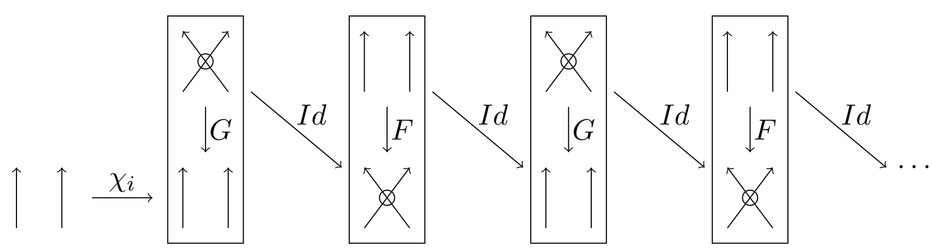


Figure 6.1: Diagrammatic representation of $P_{1,1}$.

From this presentation, we can explicitly see the filtration of $P_{1,1}$. We can compute the $Sym^2(V_{\omega_1})$-colored unknot by taking the Hochschild homology of $P_{1,1}$ and taking the homology of the resulting complex. The computations for the $(2, 2n+1)$-torus knot in Proposition 5.5.1, tells us that the Poincare series of $H(\mathrm{HH}(P_{1,1}))$ is

$$\mathcal{P}(P_{1,1}) = \frac{1+aq^2}{1-q^2}\sum_{i=0}^{\infty}\left(s^{2k+1}+ats^{2k+2}\right),\ s = tb^{-1}q^{-2}.$$

6.2 Filtered HOMFLY-PT Homology of Transverse Links

The validity of Conjecture 5.3.1 would give us an invariant of transverse links as well. In this section we will introduce the notion of a transverse link in $\mathbb{R}^3$ and then describe how our construction gives an invariant of transverse links. Our approach for basic contact topology and transversal knots will follow [13].

Suppose we consider $\mathbb{R}^3$ with some orientation, then a *contact structure* on $\mathbb{R}^3$ is a completely non-integrable plane field ξ in $T\mathbb{R}^3$. We may describe $\xi_x \subset T_x\mathbb{R}^3$ as the kernel of some 1-form α. The completely non-integrable condition can be seen as $\alpha \wedge d\alpha \neq 0$. where $\alpha \wedge d\alpha$ gives the orientation on $\mathbb{R}^3$.

Example 6.2.1. The simplest example of a contact structure on $\mathbb{R}^3$, with Cartesian coordinates (x, y, z), is given by

$$\xi_{std} = span\left\{\frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\right\}.$$

We call this contact structure the *standard contract structure* on $\mathbb{R}^3$. ξ_{std} can be seen also as the kernel of the 1-form $\alpha = dz - ydx$, and can easily be verified to be a contact structure.

Definition 6.2.2. Suppose $\mathbb{R}^3$ has contact structure ξ. A *transverse knot* K is an embedded S^1 such that K is always transverse to ξ. That is,

$$T_xK + \xi_x = T_x\mathbb{R}^3,\ x \in K.$$

Definition 6.2.3. Two transverse knots are said to be *transversely isotopic* if they are isotopic via an isotopy through transverse knots.

Up until this point we have always considered knots and links as being closures of braids. We want to know if there is a similar result for transverse knots as well. Bennequin proved the following result along those lines.

Theorem 6.2.4 (Bennequin, [7]). *Any transverse knot in $\mathbb{R}^3$ with the standard contact structure is transversely isotopic to a closed braid.*

Finally we have an analogue of the Markov theorem for transverse knots as well.

Theorem 6.2.5 (Orevkov and Shevchishin, [33]). *Two braids represent the same transverse knot if and only if they are related by braid isotopy, conjugation in the braid group (Markov Type I) and positive stabilization.*

Finally, we can address how our construction relates to transverse knots. Recall in the proof of Theorem 5.3.4, we were able to prove that positive stabilization is a filtered homotopy equivalence, but negative stabilization was not a filtered homotopy equivalence though it induced a filtered isomorphism on $\mathcal{H}_F(\beta)$. Let us reconsider the case of negative stabilization. Recall that $\mathcal{C}\ell_{n+1}(\sigma_n^{-1})$ is the complex

$$\mathbf{t}^{-1}Cone\left(R' \otimes_{\mathbb{Q}} \mathbb{Q}[x_{n+1}, \theta_{n+1}] \xrightarrow{\tilde{m}} \mathbf{bq}^2 R'\right) \xrightarrow{\chi_o} R' \otimes_{\mathbb{Q}} \mathbb{Q}[x_{n+1}, \theta_{n+1}].$$

We proved that, up to a grading shift, this complex was homotopy equivalent to R' in homological degree 0. However, $F_1(\mathcal{C}\ell_{n+1}(\sigma_n^{-1}))) = \mathbf{t}^{-1}\mathbf{bq}^2 R'$ and $F_1(R') = 0$. Let $\mathcal{H}_F^p(\beta) = H(\widetilde{\mathrm{HH}}(F_p(\widetilde{\mathcal{G}}(\beta))))$. Therefore, $\mathcal{H}_F^p(\beta)$ is *not* isomorphic to $\mathcal{H}_F^p(\beta\sigma_n^{-1})$ for all p. Therefore the filtered subcomplexes of $\mathcal{G}(\beta)$ are not invariant under negative stabilization. However this is exactly the Markov move we leave out when we want to prove that two braids represent transversely isotopic braids. Therefore the following conjecture follows immediately from Conjecture 5.3.1

Conjecture 6.2.1. *Suppose β is a braid representative of the transverse link L. Then $\mathcal{H}_F^p(\beta)$ is a transverse link invariant for all p.*

Theorem 5.3.5 will give us bounds on how the b-grading will change under Reidemeister III isotopy for generators of $\mathcal{H}_F^p(\beta)$ as well. So in the case that Conjecture 5.3.1 is false, then we still have interesting information for transverse links. Future work will explore this family of invariants and see what new information can be obtained by studying these invariants.

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